PYU33P15: Atomic Physics
Problem Set 1 due 26/11/2021

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Q1.

$$\frac{\mu\, v^2}{r} = \frac{Z\, e^2}{4\,\pi\,\varepsilon_0\, r^2} \qquad \text{(equating the centripetal and Coulomb forces)}$$

$$\begin{aligned}
E &= K + U && \text{(total energy expression)}\\
&= \frac{\mu\, v^2}{2} - \int \frac{Z\, e^2}{4\,\pi,\varepsilon_0\, r^2}\,(-\hat{r}\cdot\hat{r})\, dr && \text{(substituting } K \text{ and } U\text{)}\\
&= \frac{\mu\, v^2}{r}\,\frac{r}{2} + \frac{Z\, e^2}{4\,\pi\,\varepsilon_0}\int \frac{dr}{r^2} && \text{(rewriting and simplifying)}\\
&= \frac{Z\, e^2}{8\,\pi\,\varepsilon_0\, r} - \frac{Z\, e^2}{4\,\pi\,\varepsilon_0\, r} && \text{(substituting and computing the integral)}\\
&= -\frac{Z\, e^2}{8\,\pi\,\varepsilon_0\, r}
\end{aligned}$$

$$\begin{aligned}
\mu\, v\, r &= n\,\hbar,\ n = 1, 2, \ldots && \text{(assuming quantised angular momentum in integer values of } \hbar\text{)}\\
\mu^2\, v^2\, r^2 &= n^2\,\hbar^2 && \text{(squaring both sides)}\\
\frac{\mu\, v^2}{r}\,\mu\, r^3 &= n^2\,\hbar^2 && \text{(rewriting)}\\
\frac{Z\, e^2}{4\,\pi\,\varepsilon_0\, r^2}\,\mu\, r^3 &= n^2\,\hbar^2 && \text{(substituting)}\\
r &= \frac{4\,\pi\,\varepsilon_0\,\hbar^2\, n^2}{\mu\, Z\, e^2} && \text{(rearranging for } r\text{)}
\end{aligned}$$

$$\begin{aligned}
E &= -\frac{Z\, e^2}{8\,\pi\,\varepsilon_0}\,\frac{\mu\, Z\, e^2}{4\,\pi\,\varepsilon_0\,\hbar^2\, n^2} && \text{(substituting } r \text{ into the expression for } E\text{)}\\
&= -\frac{\mu\, Z^2\, e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2\, n^2} && \text{(simplifying)}\\
\Delta E_{i\to f} &= E_f - E_i\\
&= \frac{\mu\, Z^2\, e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right)\\
\Delta E_{2\to 1} &= \frac{\mu\, Z^2\, e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2}\left(\frac{1}{2^2} - \frac{1}{1^2}\right) && \text{(substituting } n_i = 2,\ n_f = 1\text{)}\\
&= -\frac{3\,\mu\, Z^2\, e^4}{128\,\pi^2\,\varepsilon_0^2\,\hbar^2}
\end{aligned}$$

$$
\begin{aligned}
\lambda_{2\to1} &= \frac{h\,c}{-\Delta E_{2\to1}} && \left(\text{using } E = \tfrac{h\,c}{\lambda}\right)\\
&= 2\,\pi\,\hbar\,c\,\frac{128\,\pi^2\,\varepsilon_0^2\,\hbar^2}{3\,\mu\,Z^2\,e^4} && (\text{substituting})\\
&= \frac{256\,\pi^3\,\varepsilon_0^2\,\hbar^3\,c}{3\,\mu\,Z^2\,e^4} && (\text{simplifying})
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{\min} \leq &\quad \lambda_{2\to1} \quad \leq \lambda_{\max}\\
\frac{1}{\lambda_{\min}} \geq &\quad \frac{3\,\mu\,Z^2\,e^4}{256\,\pi^3\,\varepsilon_0^2\,\hbar^3\,c} \quad \geq \frac{1}{\lambda_{\max}}\\
\frac{256\,\pi^3\,\varepsilon_0^2\,\hbar^3\,c}{3\,\mu\,e^4\,\lambda_{\min}} \geq &\quad Z^2 \quad \geq \frac{256\,\pi^3\,\varepsilon_0^2\,\hbar^3\,c}{3\,\mu\,e^4\,\lambda_{\max}}\\
\frac{16\,\pi^{\frac12}\,\varepsilon_0\,\hbar^{\frac32}\,c^{\frac12}}{3^{\frac12}\,\mu^{\frac12}\,e^2\,\lambda_{\min}^{\frac12}} \geq &\quad Z \quad \geq \frac{16\,\pi^{\frac12}\,\varepsilon_0\,\hbar^{\frac32}\,c^{\frac12}}{3^{\frac12}\,\mu^{\frac12}\,e^2\,\lambda_{\max}^{\frac12}}\\
142.3 \geq &\quad Z \quad \geq 3.1,
\end{aligned}
$$

where $\mu \approx m_e$ and $\lambda_{\min}, \lambda_{\max} = 0.06$ Å, 125 Å. Since Z is explicitly an integer value, Z ranges from 4 to 142.

Q2.

$$
\begin{aligned}
\Delta E_{i\to f} &= \frac{\mu\,e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) && (\text{as before with } Z = 1)\\
\Delta E_{3\to2} &= \frac{\mu\,e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2}\left(\frac{1}{3^2} - \frac{1}{2^2}\right) && (\text{substituting } n_i = 3,\ n_f = 2)\\
&= \frac{5\,\mu\,e^4}{1152\,\pi^2\,\varepsilon_0^2\,\hbar^2}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{3\to2} &= \frac{h\,c}{-\Delta E_{3\to2}}\\
&= \frac{2304\,\pi^3\,\varepsilon_0^2\,\hbar^3\,c}{5\,\mu\,e^4}
\end{aligned}
$$

$$
\begin{aligned}
\Delta\lambda_{3\to2} &= \left|\frac{2304\,\pi^3\,\varepsilon_0^2\,\hbar^3\,c}{5\,\mu_1\,e^4} - \frac{2304\,\pi^3\,\varepsilon_0^2\,\hbar^3\,c}{5\,\mu_2\,e^4}\right|\\
&= \frac{2304\,\pi^3\,\varepsilon_0^2\,\hbar^3\,c}{5\,e^4}\left|\frac{m_e + m_p}{m_e\,m_p} - \frac{m_e + m_p + m_n}{m_e\,(m_p + m_n)}\right|
\end{aligned}
$$

(substituting μ_1, μ_2 as the reduced mass of the hydrogen and deuterium electron)

$$\approx 0.179 \text{ nm}$$

Q3.

(a)

$$
\Delta E_{i\to f} = \frac{9\,\mu\,e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) \qquad (\text{as before with } Z = 3)
$$

$$
\begin{aligned}
|\Delta E_{2\to 1}| \leq &\quad |\Delta E_{\text{Lyman}}| \quad \leq |\Delta E_{\infty\to 1}| \\
\left|\frac{9\,\mu\,e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2}\left(\frac{1}{2^2}-\frac{1}{1^2}\right)\right| \leq &\quad |\Delta E_{\text{Lyman}}| \quad \leq \left|\frac{9\,\mu\,e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2}\left(0-\frac{1}{1^2}\right)\right| \\
\frac{27\,\mu\,e^4}{128\,\pi^2\,\varepsilon_0^2\,\hbar^2} \leq &\quad |\Delta E_{\text{Lyman}}| \quad \leq \frac{9\,\mu\,e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2} \\
2\,\pi\,\hbar\,c\,\frac{128\,\pi^2\,\varepsilon_0^2\,\hbar^2}{27\,\mu\,e^4} \geq &\quad \frac{2\,\pi\,\hbar\,c}{|\Delta E_{\text{Lyman}}|} \quad \geq 2\,\pi\,\hbar\,c\,\frac{32\,\pi^2\,\varepsilon_0^2\,\hbar^2}{9\,\mu\,e^4} \\
\frac{256\,\pi^3\,\varepsilon_0^2\,\hbar^3\,c}{27\,\mu\,e^4} \geq &\quad \lambda_{\text{Lyman}} \quad \geq \frac{64\,\pi^3\,\varepsilon_0^2\,\hbar^3\,c}{9\,\mu\,e^4},
\end{aligned}
$$

and so (assuming there are 3 protons and 4 neutrons in an atom of doubly ionised lithium and so $\mu = \frac{m_e(3m_p+4m_n)}{m_e+3m_p+4m_n}$), the wavelength ranges from 10.126 nm to 13.501 nm.

(b)

$$
E = -\frac{9\,\mu\,e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2} \qquad \text{(as before with } Z=3,\ n=1\text{)}
$$

$$
|E| \approx 1.962\times 10^{-17}\text{ J} \approx 122.422\text{ eV}
$$

Q4.

(a)

$$
\begin{aligned}
\vec{F}(r) &= -\frac{G\,m\,M}{r^2}\,\hat{r} \\
U(r) &= -\int \vec{F}\cdot d\vec{r} \\
&= \int \frac{G\,m\,M}{r^2}\,dr \\
&= -\frac{G\,m\,M}{r}
\end{aligned}
$$

(b)

If we compare the potential energy function for the earth-sun system to that of the Hydrogen atom, we can convert expressions such as the Bohr radius and Bohr formula as needed for the earth-sun system.

$$
U_{\text{hydrogen}}(r) \to U_{\text{earth-sun}}(r) \implies -\frac{e^2}{4\,\pi\,\varepsilon_0\,r} \to -\frac{G\,m\,M}{r} \implies \frac{e^2}{4\,\pi\,\varepsilon_0} \to G\,m\,M
$$

$$
a_0 = \frac{4\,\pi\,\varepsilon_0\,\hbar^2}{\mu\,e^2} = \left(\frac{\mu}{\hbar^2}\,\frac{e^2}{4\,\pi\,\varepsilon_0}\right)^{-1} \to \left(\frac{\mu}{\hbar^2}\,G\,m\,M\right)^{-1} = \frac{\hbar^2}{G\,m\,M\,\mu} = a_g
$$

Thus the gravitational Bohr radius is given by $a_g = \frac{\hbar^2}{G\,m\,M\,\mu} \approx 2.349\times 10^{-138}$.

$$
E_{n,\text{hydrogen}} = -\frac{\mu\,e^4}{32\,\pi^2\,\varepsilon_0^2\,\hbar^2\,n^2} = -\frac{e^2}{4\,\pi\,\varepsilon_0}\left(\frac{4\,\pi\,\varepsilon_0\,\hbar^2}{\mu\,e^2}\right)^{-1}\frac{1}{2\,n^2} \to -G\,m\,M\,a_g^{-1}\,\frac{1}{2\,n^2} = -\frac{G\,m\,M}{2\,a_g\,n^2} = E_{n,\text{gravitational}}
$$

Equating this expression of the energy of the earth-sun system to the classical result $-\frac{G\,m\,M}{2\,r_0}$ gives us $a_g\,n^2 = r_0$, i.e. $n = \sqrt{\frac{r_0}{a_g}}$.

(c)

$$
\begin{aligned}
\Delta E_{i\to f} &= \frac{G\,m\,M}{2\,a_g}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) \\
\Delta E_{n\to n-1} &= \frac{G\,m\,M}{2\,a_g}\left(\frac{1}{n^2} - \frac{1}{(n-1)^2}\right) \\
&= \frac{G\,m\,M}{2\,a_g\,n^2}\left(1 - \frac{1}{(1-\frac{1}{n})^2}\right) \\
&= \frac{G\,m\,M}{2\,a_g\,n^2}\left(1 - 1 - \frac{2}{n} - \frac{3}{n^2} - \frac{4}{n^3} - \dots\right) && \left(\text{expanding } \tfrac{1}{(1-\frac{1}{n})^2} \text{ in a Taylor series about } \infty\right) \\
&\approx -\frac{G\,m\,M}{a_g\,n^3} && (\text{neglecting terms of } n^{-2} \text{ and lower}) \\
&= -G\,m\,M\,\frac{G\,m\,M\,\mu}{\hbar^2}\,\frac{1}{r_0^{\frac{3}{2}}}\,\frac{\hbar^3}{(G\,m\,M\,\mu)^{\frac{3}{2}}} && (\text{substituting } n \text{ and } a_g) \\
&= \frac{\hbar\sqrt{G\,m\,M}}{r_0^{\frac{3}{2}}\sqrt{\mu}} \\
|\Delta E_{n\to n-1}| &\approx 2.1\times 10^{-41}\ \text{J}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{n\to n-1} &= \frac{2\,\pi\,\hbar\,c}{-\Delta E_{n\to n-1}} \\
&\approx \frac{2\,\pi\,\hbar^2\,n^3}{G^2\,m^2\,M^2\,\mu} \\
&\approx 0.9999 \text{ lightyears}
\end{aligned}
$$

Q5.

$$
\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\,Y_l^m(\theta,\phi)
$$

$$
\begin{aligned}
R_{nl}(r) &= \frac{1}{r}\,(\kappa\,r)^{l+1}\,e^{-\kappa\,r}\,\nu(\kappa\,r) \\
R_{1,0}(r) &= \frac{1}{r}\,\kappa\,r\,e^{-\kappa\,r}\,c_0 \\
&= c_0\,\kappa\,e^{-\kappa\,r}
\end{aligned}
$$

$$
\begin{aligned}
\kappa &= \frac{\sqrt{-2\,m\,E}}{\hbar} \\
&= \frac{1}{\hbar}\sqrt{\frac{2\,m^2}{2\,\hbar^2}\left(\frac{e^2}{4\,\pi\,\varepsilon_0}\right)^2} && (\text{for } n = 1) \\
&= \frac{m\,e^2}{4\,\pi\,\varepsilon_0\,\hbar^2} \\
&= \frac{1}{a_0} \\
\implies R_{1,0}(r) &= \frac{c_0}{a_0}\,e^{-\frac{r}{a_0}}
\end{aligned}
$$

$$
\begin{aligned}
1 &= \int_0^\infty \left|R_{1,0}(r)\right|^2 \, dr \\
&= \int_0^\infty \frac{|c_0|^2}{a_0^2} \, e^{-\frac{2r}{a_0}} \\
&= \frac{|c_0|^2}{a_0^2} \, \frac{a_0^3}{4} \\
\implies |c_0| &= \frac{2}{\sqrt{a_0}}
\end{aligned}
$$

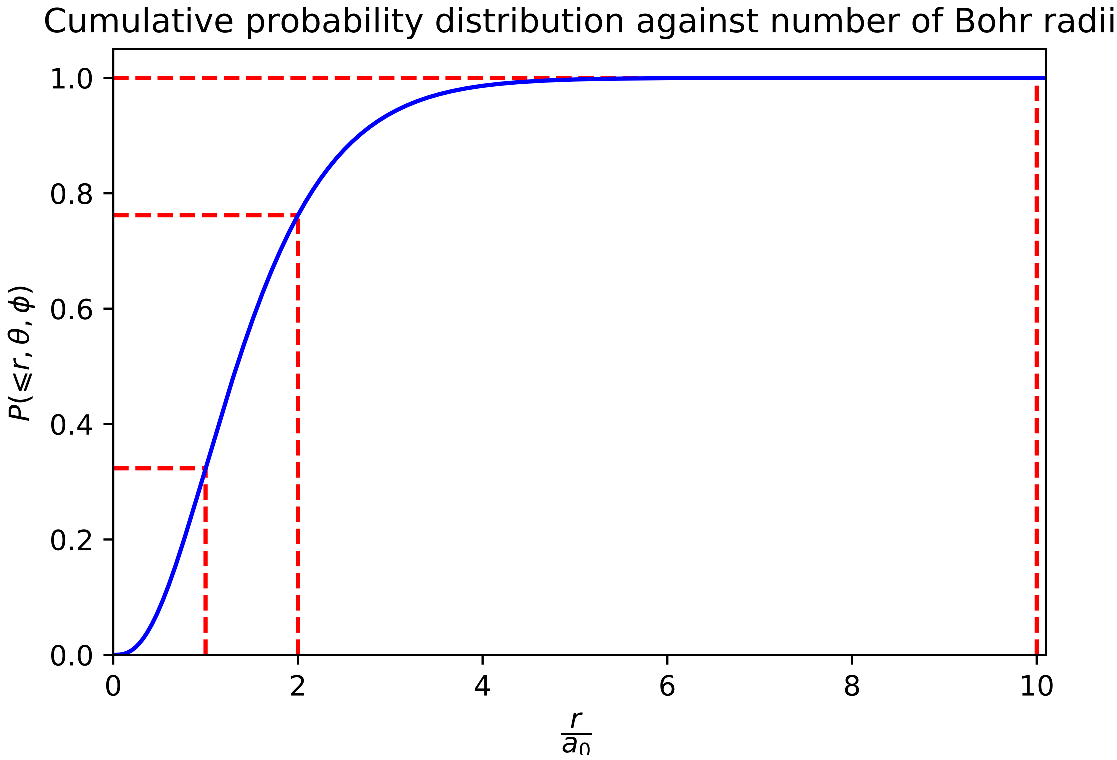
$$
\begin{aligned}
Y_l^m &= (-1)^m \sqrt{\frac{2\,l+1}{4\,\pi} \, \frac{(l-|m|)^2}{(l+|m|)^2}} \, e^{i\,m\,\phi} \, P_l^m(\cos\theta) \\
Y_0^0 &= \frac{1}{\sqrt{4\,\pi}} = \frac{1}{2\sqrt{\pi}} \\
\implies \psi_{1,0,0}(r,\theta,\phi) &= \frac{c_0}{2\,a_0\sqrt{\pi}} \, e^{-\frac{r}{a_0}}
\end{aligned}
$$

$$
\begin{aligned}
P(\le r,\theta,\phi) &= \int_0^r \left|\psi_{1,0,0}(r,\theta,\phi)\right|^2 \, dV \\
&= \int_0^r \frac{|c_0|^2}{4\,a_0^2\,\pi} \, e^{-\frac{2r}{a_0}} \, 4\,\pi^2 \, r \, dr \\
&\text{(substituting } \psi_{1,0,0} \text{ and } dV = 4\,\pi\,r^2\,dr\text{, i.e. volume of shell of radius } r \text{ and thickness } dr\text{)} \\
&= \int_0^r \frac{4\,r^2}{a_0^3} \, e^{-\frac{2r}{a_0}} \, dr && \text{(substituting } |c_0|\text{)} \\
&= 4 \int_0^{\frac{r}{a_0}} \rho^2 \, e^{-\rho} \, d\rho && (r = a_0\,\rho,\ dr = a_0\,d\rho) \\
&= 4 \left(-\frac{\rho^2\,e^{-2\rho}}{2} \Bigg|_0^{\frac{r}{a_0}} + \int_0^{\frac{r}{a_0}} \rho\,e^{-2\rho} \, d\rho \right) && \text{(integration by parts with } u = \rho^2,\ dv = e^{-2\rho}\,d\rho\text{)} \\
&= 4 \left(-\frac{r^2\,e^{-\frac{2r}{a_0}}}{2\,a_0^2} + 0 - \frac{\rho\,e^{-2\rho}}{2} \Bigg|_0^{\frac{r}{a_0}} + \frac{1}{2} \int_0^{\frac{r}{a_0}} e^{-2\rho} \, d\rho \right) \\
& && \text{(integration by parts with } u = \rho,\ dv = e^{-2\rho}\,d\rho\text{)} \\
&= 4 \left(-\frac{r^2\,e^{-\frac{2r}{a_0}}}{2\,a_0^2} - \frac{r\,e^{-\frac{2r}{a_0}}}{2\,a_0} - \frac{e^{-2\rho}}{4} \Bigg|_0^{\frac{r}{a_0}} \right) \\
&= 4 \left(-\frac{r^2\,e^{-\frac{2r}{a_0}}}{2\,a_0^2} - \frac{r\,e^{-\frac{2r}{a_0}}}{2\,a_0} - \frac{e^{-\frac{2r}{a_0}}}{4} + \frac{1}{4} \right) \\
&= 1 - e^{-\frac{2r}{a_0}} \left(1 + \frac{2\,r}{a_0} \left(1 + \frac{r}{a_0} \right) \right)
\end{aligned}
$$

$$
\begin{aligned}
P(\le a_0,\theta,\phi) &= 1 - e^{-\frac{2a_0}{a_0}} \left(1 + \frac{2\,a_0}{a_0} \left(1 + \frac{a_0}{a_0} \right) \right) \\
&= 1 - \frac{5}{e^2} \approx 0.323
\end{aligned}
$$

$$
\begin{aligned}
P(\le 2\,a_0,\theta,\phi) &= 1 - e^{-\frac{4a_0}{a_0}} \left(1 + \frac{4\,a_0}{a_0} \left(1 + \frac{2a_0}{a_0} \right) \right) \\
&= 1 - \frac{13}{e^4} \approx 0.762
\end{aligned}
$$

$$
\begin{aligned}
P(\le 10\,a_0,\theta,\phi) &= 1 - e^{-\frac{20a_0}{a_0}} \left(1 + \frac{20\,a_0}{a_0} \left(1 + \frac{10\,a_0}{a_0} \right) \right) \\
&= 1 - \frac{221}{e^{20}} \approx 0.9999995
\end{aligned}
$$


Cumulative probability distribution against number of Bohr radii
$P(\leqslant r, \theta, \phi)$
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
$\frac{r}{a_0}$

Q6.

$$
\begin{aligned}
[H, L_x]\, f =& \left[\frac{\vec{P}^2}{2m} + V(\vec{r})\,, L_x\right] f \\
=& \frac{1}{2m}\left[\vec{P}^2, L_x\right] f + \left[V(\vec{r})\,, L_x\right] f \\
=& \frac{1}{2m}\left(\vec{P}\left[P_y\,\hat{x} + P_y\,\hat{y} + P_z\,\hat{z}, Y\,P_z - Z\,P_y\right] + \left[P_y\,\hat{x} + P_y\,\hat{y} + P_z\,\hat{z}, Y\,P_z - Z\,P_y\right]\vec{P}\right) f \\
&+ \left[V(\vec{r})\,, Y\,P_z - Z\,P_y\right] f \\
=& \frac{1}{2m}\left(\vec{P}\left(\left[P_y, Y\,P_z\right]\hat{y} + \left[P_z, -Z\,P_y\right]\hat{z}\right) + \left(\left[P_y, Y\,P_z\right]\hat{y} + \left[P_z, -Z\,P_y\right]\hat{z}\right)\vec{P}\right) f \\
&+ \left[V(\vec{r})\,, Y\right] P_z\, f + Y\left[V(\vec{r})\,, P_z\right] f - \left[V(\vec{r})\,, Z\right] P_y\, f - Z\left[V(\vec{r})\,, P_y\right] f \\
=& \frac{1}{2m}\left(\vec{P}\left(-i\,\hbar\,P_z\,\hat{y} + i\,\hbar\,P_y\,\hat{z}\right) + \left(-i\,\hbar\,P_z\,\hat{y} + i\,\hbar\,P_y\,\hat{z}\right)\vec{P}\right) f - i\,\hbar\left(V(\vec{r})\,Y\,\frac{\partial}{\partial Z} f - Y\,V(\vec{r})\,\frac{\partial}{\partial Z} f\right. \\
&\left. + Y\,V(\vec{r})\,\frac{\partial}{\partial Z} f - Y\,\frac{\partial}{\partial Z} V(\vec{r})\, f - V(\vec{r})\,Z\,\frac{\partial}{\partial Y} f + Z\,V(\vec{r})\,\frac{\partial}{\partial Y} f - Z\,V(\vec{r})\,\frac{\partial}{\partial Y} f + Z\,\frac{\partial}{\partial Y} V(\vec{r})\, f\right) \\
=& \frac{i\,\hbar}{2m}\left(-P_y\,P_z + P_z\,P_y - P_y\,P_z + P_z\,P_y\right) f - i\,\hbar\left(V(\vec{r})\,Y\,\frac{\partial f}{\partial Z} - Y\,V(\vec{r})\,\frac{\partial f}{\partial Z} + Y\,V(\vec{r})\,\frac{\partial f}{\partial Z}\right. \\
&\left. - Y\,\frac{\partial V(\vec{r})}{\partial Z}\, f - Y\,V(\vec{r})\,\frac{\partial f}{\partial Z} - V(\vec{r})\,Z\,\frac{\partial f}{\partial Y} + Z\,V(\vec{r})\,\frac{\partial f}{\partial Y} - Z\,V(\vec{r})\,\frac{\partial f}{\partial Y} + Z\,\frac{\partial V(\vec{r})}{\partial Y}\, f + Z\,V(\vec{r})\,\frac{\partial f}{\partial Y}\right) \\
=& 0 - i\,\hbar\left(V(\vec{r})\,Y\,\frac{\partial f}{\partial Z} - Y\,\frac{\partial V(\vec{r})}{\partial Z}\, f - Y\,V(\vec{r})\,\frac{\partial f}{\partial Z} - V(\vec{r})\,Z\,\frac{\partial f}{\partial Y} + Z\,\frac{\partial V(\vec{r})}{\partial Y}\, f + Z\,V(\vec{r})\,\frac{\partial f}{\partial Y}\right) \\
[H, L_x] =& -i\,\hbar\left(-Y\,\frac{\partial V(\vec{r})}{\partial Z} + Z\,\frac{\partial V(\vec{r})}{\partial Y}\right) \\
=& i\,\hbar\left(\vec{R}\times\nabla V\right)_x
\end{aligned}
$$

Similarly, $[H, L_y] = i\,\hbar\left(\vec{R}\times\nabla V\right)_y$ and $[H, L_z] = i\,\hbar\left(\vec{R}\times\nabla V\right)_z$, and so

$$
\begin{aligned}
\left[H, \vec{L}\right] &= i\,\hbar\,\vec{R}\times\nabla V \\
\frac{d\left\langle\vec{L}\right\rangle}{dt} &= \frac{i}{\hbar}\left\langle\left[H, \vec{L}\right]\right\rangle + \left\langle\frac{\partial\vec{L}}{\partial t}\right\rangle \\
&= -\vec{R}\times\nabla V + 0 \\
&= \vec{R}\times\left(-\nabla V\right)
\end{aligned}
$$

$$
\begin{aligned}
V(\vec{r}) &= V(r) \\
\implies \nabla V &= \frac{\partial V(r)}{\partial r}\,\hat{r} + \frac{1}{r}\,\frac{\partial V(r)}{\partial\theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\,\frac{\partial V(r)}{\partial\phi}\,\hat{\phi} \\
&= \frac{\partial V(r)}{\partial r}\,\hat{r} + 0 \\
\implies \frac{d\left\langle\vec{L}\right\rangle}{dt} &= \vec{R}\times\frac{\partial V(r)}{\partial r}\,\hat{r} \\
&= \left(\vec{R}\times\hat{r}\right)\left(-\frac{\partial V(r)}{\partial r}\right) \\
&= 0
\end{aligned}
$$