MAU23203: Analysis in Several Real Variables
Homework 2 due 30/11/2021

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Name:	Ruaidhrí Campion
Student number:	19333850
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1.

The limit of $f(\mathbf{x})$ is 0 as $\mathbf{x}$ tends to $\mathbf{p}$, and so given any positive ε there exists a δ such that $|f(\mathbf{x})-\mathbf{0}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x}-\mathbf{p}| < \delta$. Since $|h(\mathbf{x})| \leqslant M$ for all $\mathbf{x} \in X$, then $|h(\mathbf{x})\, f(\mathbf{x}) - \mathbf{0}| = |h(\mathbf{x})|\, |f(\mathbf{x})| \leqslant M\, |f(\mathbf{x})|$. Then, if $|f(\mathbf{x})| = |f(\mathbf{x})-\mathbf{0}| < \varepsilon$, it is true that $|h(\mathbf{x})\, f(\mathbf{x})-\mathbf{0}| \leqslant M\, |f(\mathbf{x})| < M\,\varepsilon$. Therefore given any positive ε there exists a δ such that $|h(\mathbf{x})\, f(\mathbf{x}) - \mathbf{0}| < M\,\varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x}-\mathbf{p}| < \delta$. Since M and ε are both positive constants, their product $M\,\varepsilon$ is also positive. Relabelling $M\,\varepsilon \equiv \varepsilon'$, and considering any positive ε' instead of any positive ε, the statement meets the definition of the limit of $h(\mathbf{x})\, f(\mathbf{x})$ being 0 as $\mathbf{x}$ tends to $\mathbf{p}$, as required.

2.

By equating the given expression for $f(\mathbf{x})\, g(\mathbf{x})$ with the product of the given expressions for $f(\mathbf{x})$ and $g(\mathbf{x})$, an expression for $w(x)$ can be found.

$$\begin{aligned}
f(\mathbf{x})\, g(\mathbf{x}) &= \Big[f(\mathbf{p}) + (\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) + |\mathbf{x}-\mathbf{p}|\, u(\mathbf{x})\Big] \Big[g(\mathbf{p}) + (\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) + |\mathbf{x}-\mathbf{p}|\, v(\mathbf{x})\Big] \\
&= f(\mathbf{p})\, g(\mathbf{p}) + f(\mathbf{p})\, (\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) + f(\mathbf{p})\, |\mathbf{x}-\mathbf{p}|\, v(\mathbf{x}) \\
&\quad + (\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\, g(\mathbf{p}) + \Big((\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\Big) \Big((\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\Big) \\
&\quad + (\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\, |\mathbf{x}-\mathbf{p}|\, v(\mathbf{x}) + |\mathbf{x}-\mathbf{p}|\, u(\mathbf{x})\, g(\mathbf{p}) + |\mathbf{x}-\mathbf{p}|\, u(\mathbf{x})\, (\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) \\
&\quad + |\mathbf{x}-\mathbf{p}|^2\, u(\mathbf{x})\, v(\mathbf{x}) \\
f(\mathbf{x})\, g(\mathbf{x}) &= f(\mathbf{p})\, g(\mathbf{p}) + g(\mathbf{p})\, (\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) + f(\mathbf{p})\, (\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) + |\mathbf{x}-\mathbf{p}|\, w(\mathbf{x}) \\
\implies |\mathbf{x}-\mathbf{p}|\, w(\mathbf{x}) &= |\mathbf{x}-\mathbf{p}|\, f(\mathbf{p})\, v(\mathbf{x}) + \Big((\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\Big) \Big((\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\Big) \\
&\quad + |\mathbf{x}-\mathbf{p}|\, v(\mathbf{x})\, (\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) + |\mathbf{x}-\mathbf{p}|\, g(\mathbf{p})\, u(\mathbf{x}) + |\mathbf{x}-\mathbf{p}|\, u(\mathbf{x})\, (\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) \\
&\quad + |\mathbf{x}-\mathbf{p}|^2\, u(\mathbf{x})\, v(\mathbf{x})
\end{aligned}$$

Since $w(\mathbf{p}) = 0$ is known and thus does not need to be accounted for, both sides of the above expression can be divided by $|\mathbf{x}-\mathbf{p}|$ to yield $w(\mathbf{x})$ for $\mathbf{x} \neq \mathbf{p}$, i.e.

$$w(\mathbf{x}) = \begin{cases} f(\mathbf{p})\, v(\mathbf{x}) + \frac{\left((\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\right)\left((\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\right)}{|\mathbf{x}-\mathbf{p}|} + v(\mathbf{x})\, (\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) + g(\mathbf{p})\, u(\mathbf{x}) & \\ \quad + u(\mathbf{x})\, (\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) + |\mathbf{x}-\mathbf{p}|\, u(\mathbf{x})\, v(\mathbf{x}), & \mathbf{x} \neq \mathbf{p} \\ 0, & \mathbf{x} = \mathbf{p} \end{cases}$$

Taking the limit of both sides as $\mathbf{x}$ tends to $\mathbf{p}$ yields

$$\lim_{\mathbf{x}\to\mathbf{p}} w(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{p}} \Bigg(f(\mathbf{p})\, v(\mathbf{x}) + \frac{\Big((\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\Big)\Big((\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\Big)}{|\mathbf{x}-\mathbf{p}|} + v(\mathbf{x})\,(\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) + g(\mathbf{p})\, u(\mathbf{x})$$
$$+\, u(\mathbf{x})\,(\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) + |\mathbf{x}-\mathbf{p}|\, u(\mathbf{x})\, v(\mathbf{x}) \Bigg)$$

The right hand side can be split into a sum of limits if the limit of each term exists, and if this is the case, is simply equal to the sum of the limit of each term (Proposition 6.2). Since $f(\mathbf{p})$ and $g(\mathbf{p})$ are constant, and the limits of $u(\mathbf{x})$, $v(\mathbf{x})$ and $|\mathbf{x}-\mathbf{p}|$ as $\mathbf{x}$ tends to $\mathbf{p}$ are defined, then the limit of a product of any of these terms is the product of the limits of each term (Proposition 6.4). Thus,

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{p})\, v(\mathbf{x}) &= \left(\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{p})\right)\left(\lim_{\mathbf{x}\to\mathbf{p}} v(\mathbf{x})\right) \\
&= f(\mathbf{p})\,(0) \\
&= 0 \\
\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{p})\, u(\mathbf{x}) &= \left(\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{p})\right)\left(\lim_{\mathbf{x}\to\mathbf{p}} u(\mathbf{x})\right) \\
&= g(\mathbf{p})\,(0) \\
&= 0 \\
\lim_{\mathbf{x}\to\mathbf{p}} |\mathbf{x}-\mathbf{p}|\, u(\mathbf{x})\, v(\mathbf{x}) &= \left(\lim_{\mathbf{x}\to\mathbf{p}} |\mathbf{x}-\mathbf{p}|\right)\left(\lim_{\mathbf{x}\to\mathbf{p}} u(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{p}} v(\mathbf{x})\right) \\
&= (0)\,(0)\,(0) \\
&= 0
\end{aligned}$$

The dot product of $(\nabla f)_{\mathbf{p}}$ with another vector $\mathbf{q}(\mathbf{x})$ can be considered as a continuous function mapping $\mathbf{q}(\mathbf{x})$ to $\mathbb{R}$. Thus, the limit of the dot product of $(\nabla f)_{\mathbf{p}}$ with the vector $\mathbf{q}(\mathbf{x})$ can be written as the dot product of $(\nabla f)_{\mathbf{p}}$ with the limit of $\mathbf{q}(\mathbf{x})$, provided the limit of $\mathbf{q}(\mathbf{x})$ exists (Lemma 6.3). A similar consideration can be made for $(\nabla g)_{\mathbf{p}}$ instead of $(\nabla f)_{\mathbf{p}}$. Thus,

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} (\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) &= (\nabla f)_{\mathbf{p}} \cdot \lim_{\mathbf{x}\to\mathbf{p}} (\mathbf{x}-\mathbf{p}) \\
&= (\nabla f)_{\mathbf{p}} \cdot \mathbf{0} \\
&= 0 \\
\implies \lim_{\mathbf{x}\to\mathbf{p}} v(\mathbf{x})\,(\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) &= \left(\lim_{\mathbf{x}\to\mathbf{p}} v(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{p}} (\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\right) \\
&= (0)\,(0) \\
&= 0 \\
\lim_{\mathbf{x}\to\mathbf{p}} (\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) &= (\nabla g)_{\mathbf{p}} \cdot \lim_{\mathbf{x}\to\mathbf{p}} (\mathbf{x}-\mathbf{p}) \\
&= (\nabla g)_{\mathbf{p}} \cdot \mathbf{0} \\
&= 0 \\
\implies \lim_{\mathbf{x}\to\mathbf{p}} u(\mathbf{x})\,(\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) &= \left(\lim_{\mathbf{x}\to\mathbf{p}} u(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{p}} (\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})\right) \\
&= (0)\,(0) \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\frac{(\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})}{|\mathbf{x}-\mathbf{p}|} &= (\nabla g)_{\mathbf{p}} \cdot \frac{1}{|\mathbf{x}-\mathbf{p}|}(\mathbf{x}-\mathbf{p}) \\
&= (\nabla g)_{\mathbf{p}} \cdot |\mathbf{x}-\mathbf{p}| \, \hat{e}_{(\mathbf{x}-\mathbf{p})} \\
\implies \left| \frac{(\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})}{|\mathbf{x}-\mathbf{p}|} \right| &\leqslant |\mathbf{x}-\mathbf{p}| \left| (\nabla g)_{\mathbf{p}} \right| \\
\implies \lim_{\mathbf{x}\to\mathbf{p}} \left| \frac{(\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})}{|\mathbf{x}-\mathbf{p}|} \right| &\leqslant \lim_{\mathbf{x}\to\mathbf{p}} |\mathbf{x}-\mathbf{p}| \left| (\nabla g)_{\mathbf{p}} \right| \\
&\leqslant \left(\lim_{\mathbf{x}\to\mathbf{p}} |\mathbf{x}-\mathbf{p}| \right) \left(\lim_{\mathbf{x}\to\mathbf{p}} \left| (\nabla g)_{\mathbf{p}} \right| \right) \\
&\leqslant (0) \left| (\nabla g)_{\mathbf{p}} \right| \\
&\leqslant 0 \\
\implies \lim_{\mathbf{x}\to\mathbf{p}} \frac{(\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})}{|\mathbf{x}-\mathbf{p}|} &= 0 \\
\implies \lim_{\mathbf{x}\to\mathbf{p}} \frac{\left((\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) \right) \left((\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) \right)}{|\mathbf{x}-\mathbf{p}|} &= \left(\lim_{\mathbf{x}\to\mathbf{p}} (\nabla f)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p}) \right) \left(\lim_{\mathbf{x}\to\mathbf{p}} \frac{(\nabla g)_{\mathbf{p}} \cdot (\mathbf{x}-\mathbf{p})}{|\mathbf{x}-\mathbf{p}|} \right) \\
&= (0)\,(0) \\
&= 0
\end{aligned}
$$

Therefore since the limit of each term on the right hand side of the expression for $w(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ is 0, then the limit of $w(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ is 0.