MAU23203: Analysis in Several Real Variables
Homework 1 due 04/11/2021

I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at `http://www.tcd.ie/calendar`. I have also completed the Online Tutorial on avoiding plagiarism Ready Steady Write, located at `http://tcd-ie.libguides.com/plagiarism/ready-steady-write`.

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1.

(a)

$\Longrightarrow$

- Say G is closed in X, i.e. $X \setminus G$ is open in X.
- Since $X \setminus G$ is open in X, it can be written as $X \setminus G = H \cap X$ for some set H open in $\mathbb{R}^n$ (Proposition 4.5).
- By taking the complement of both sides, G can thus be written as $G = X \setminus (H \cap X)$, i.e. the set of elements of X not shared with H, which is by definition $X \setminus H$.
- $G = X \setminus H$ can also be written as $(\mathbb{R}^n \setminus H) \cap X$, i.e. the set of elements in X shared with those in $\mathbb{R}^n$ not in H.
- Since H is open in $\mathbb{R}^n$, $F = \mathbb{R}^n \setminus H$ is closed in $\mathbb{R}^n$, and thus there exists a closed set F in $\mathbb{R}^n$ for which $G = F \cap X$.

$\Longleftarrow$

- Say $G = F \cap X$ for some F closed in $\mathbb{R}^n$.
- As before, $X \setminus G = X \setminus (F \cap X)$ can be written as $X \setminus G = (\mathbb{R}^n \setminus F) \cap X$.
- Since F is closed in $\mathbb{R}^n$, $J = \mathbb{R}^n \setminus F$ is open in $\mathbb{R}^n$.
- Since there exists a set J open in $\mathbb{R}^n$ such that $X \setminus G = J \cap X$, then $X \setminus G$ is open in X (Proposition 4.5).
- By definition, G is closed in X.

(b)

- Say that a subset X of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$, and a subset G of X is closed in X.
- From (a), if a subset G of X is closed in X then there exists a closed set F in $\mathbb{R}^n$ for which $G = F \cap X$.
- Since F and X are both closed in $\mathbb{R}^n$, then G can be expressed as an intersection of closed sets in $\mathbb{R}^n$, and therefore is closed in $\mathbb{R}^n$.

(c)

- Let a subset W of Y be the set of all elements in Y that are strictly less than c.
- $X \setminus f^{-1}(W)$, the complement in X of the preimage of W under f, will thus be the set of all elements in X that map to a value greater than or equal to c, i.e. $X \setminus f^{-1}(W) = \{x \in X \mid f(x) \geqslant c\}$.
- W can be written as $(-\infty, c) \cap Y$, i.e. as an intersection of Y and an open set in $\mathbb{R}^n$, and is thus open in Y (Proposition 4.5).
- Since f is a continuous function and W is open in Y, $f^{-1}(W)$ is open in X (Proposition 5.7).
- Therefore the complement in X of this preimage is closed in X, i.e. $X \setminus f^{-1}(W) = \{x \in X \mid f(x) \geqslant c\}$ is closed in X.

2.

Say that g is continuous at $(0,0)$. Then, by definition, for any given positive real number ε there exists a positive real number δ such that $|g(x) - g((0,0))| < \varepsilon$ whenever $|x - (0,0)| < \delta$. x can be written in the form of either $x = (0,0)$ or $x = (e^{-t}\ \cos t, e^{-t}\ \sin t)$ for some $t \in \mathbb{R}$, as $x \in \mathbb{R}$. Since $|g((0,0)) - g((0,0))| < \varepsilon$ for any positive ε and $|(0,0) - (0,0)| < \delta$ for any positive δ, only the case where $x = (e^{-t}\ \cos t, e^{-t}\ \sin t)$, needs to be considered. Thus if g is continuous at $(0,0)$, then for any positive real number ε there exists a positive real number δ such that $|g((e^{-t}\ \cos t, e^{-t}\ \sin t)) - g(0,0)| < \varepsilon$ whenever $0 < |\,(e^{-t}\ \cos t, e^{-t}\ \sin t) - (0,0)| < \delta$. Thus, from the definition of a limit of a function, the limit of $g((e^{-t}\ \cos t, e^{-t}\ \sin t))$ as $(e^{-t}\ \cos t, e^{-t}\ \sin t)$ tends to $(0,0)$ is $g(0,0)$. Therefore, since $f(t) = g((e^{-t}\ \cos t, e^{-t}\ \sin t))$, $g(0,0) = v$, and $\lim_{t\to\infty} e^{-t}\ \cos t = \lim_{t\to\infty} e^{-t}\ \sin t = 0$, if g is continuous at $(0,0)$, then $\lim_{t\to\infty} f(t) = v$.

Say that $\lim_{t\to\infty} f(t) = v$. By definition, this means that for any given positive real number ε there exists a real number L such that $|f(t) - v| < \varepsilon$ whenever $t > L$. From substitution, this is equivalent to stating that for any given positive real number ε there exists a real number L such that $|g((e^{-t}\ \cos t, e^{-t}\ \sin t)) - g(0,0)| < \varepsilon$ whenever $t > L$. The condition "whenever $t > L$" can also be written as "whenever $|\,(e^{-t}\ \cos t, e^{-t}\ \sin t)\,| < |\,\big(e^{-L}\ \cos t, e^{-L}\ \sin t\big)\,|$". By labelling $\delta = |\,\big(e^{-L}\ \cos t, e^{-L}\ \sin t\big)\,|$ it is now true that for any given positive real number ε there exists a positive real number δ such that $|g((e^{-t}\ \cos t, e^{-t}\ \sin t)) - g(0,0)| < \varepsilon$ whenever $|\,(e^{-t}\ \cos t, e^{-t}\ \sin t)\,| < \delta$. Since x can be written in the form of either $x = (0,0)$ or $x = (e^{-t}\ \cos t, e^{-t}\ \sin t)$, and for any positive ε and δ, $|g(0,0) - g(0,0)| < \varepsilon$ and $|(0,0) - (0,0)| < \delta$ are true, it is thus true that, for any given positive real number ε there exists a positive real number δ such that $|g(x) - g(0,0)| < \varepsilon$ whenever $|x - (0,0)| < \delta$. Therefore, by definition, if $\lim_{t\to\infty} f(t) = v$, then g is continuous at $(0,0)$.