MAU11102: Linear Algebra II
Homework 9 due 03/04/2020

Ruaidhrí Campion
19333850
Theoretical Physics

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Problem 1

$$\begin{aligned}
\chi_A(t) &= \det(tI - A) \\
&= \begin{vmatrix} t-4 & 2 & -2 \\ 2 & t-1 & 1 \\ -2 & 1 & t-1 \end{vmatrix} \\
&= (t-4)((t-1)^2 - 1) - 2(2(t-1)+2) - 2(2+2(t-1)) \\
&= (t-4)(t^2 - 2t) - 4t - 4t \\
&= t^3 - 6t^2 = t^2(t-6)
\end{aligned}$$

$$\chi_A(\lambda) = 0 \Rightarrow \lambda_1,\ \lambda_2 = 0,\ \lambda_3 = 6$$

$$(A - \lambda I)v = 0$$

$$(A - 0I)v = 0 \Rightarrow \left(\begin{array}{ccc|c} 4 & -2 & 2 & 0 \\ -2 & 1 & -1 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right) \begin{array}{c} r_1 + 2r_2 \\ r_3 + r_2 \end{array} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$v_3 = -2v_1 + v_2 \Rightarrow v = \begin{pmatrix} v_1 \\ v_2 \\ -2v_1 + v_2 \end{pmatrix}$$

$$\Rightarrow v_{\lambda_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix},\ v_{\lambda_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$(A - 6I)v = 0 \Rightarrow \left(\begin{array}{ccc|c} -2 & -2 & 2 & 0 \\ -2 & -5 & -1 & 0 \\ 2 & -1 & -5 & 0 \end{array}\right) \begin{array}{c} r_2 - r_1 \\ r_3 + r_1 \end{array} \left(\begin{array}{ccc|c} -2 & -2 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right)$$

$$\begin{array}{c} -\frac{1}{2}r_1 \\ -\frac{1}{3}r_2 \\ -\frac{1}{6}r_3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \begin{array}{c} r_1 - r_2 \\ r_3 - r_2 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} v_1 = 2v_3 \\ v_2 = -v_3 \end{array} \Rightarrow v = \begin{pmatrix} 2v_3 \\ -v_3 \\ v_3 \end{pmatrix}$$

$$\Rightarrow v_{\lambda_3} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

v_{λ_3} is orthogonal to both v_{λ_1} and v_{λ_2}, but v_{λ_1} and v_{λ_2} are not orthogonal. We will use the Gram-Schmidt procedure to change v_{λ_1}, v_{λ_2} and v_{λ_3} into orthonormal vectors.

$$u_1 = v_{\lambda_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \; w_1 = \frac{u_1}{||u_1||} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$u_2 = v_{\lambda_2} - (v_{\lambda_2}, w_1)w_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{\sqrt{5}} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

$$w_2 = \frac{u_2}{||u_2||} = \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

$$v_{\lambda_3} \cdot w_1 = v_{\lambda_3} \cdot w_2 = 0 \Rightarrow u_3 = v_{\lambda_3} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \; w_3 = \frac{u_3}{||u_3||} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$M = \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \middle| \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \middle| \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Check that M is orthogonal and that $M^T AM$ is a diagonal matrix

$$\begin{aligned} M^T M &= \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \Rightarrow \text{orthogonal} \end{aligned}$$

$$\begin{aligned} M^T AM &= \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{12}{\sqrt{6}} & -\frac{6}{\sqrt{6}} & \frac{6}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \end{aligned}$$

Problem 2

Let v_3 be the third column of the matrix M. v_1 and v_2 are orthonormal and so M is orthogonal if v_3 is orthonormal to both v_1 and v_2.

$$\Rightarrow v_1 \cdot v_3 = 0,\ v_2 \cdot v_3 = 0$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) r_1 - r_2 \left(\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right)$$

$$\begin{array}{c} a = c \\ b = -2c \end{array} \Rightarrow v_3 = c \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$||v_3|| = 1 \Rightarrow v_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$M = \left(\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \middle| \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Check that M is orthogonal

$$\begin{aligned} M^T M &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \Rightarrow \text{orthogonal} \end{aligned}$$

Problem 3

$$A : V \to V,\ A^* : V \to V$$

$$(Av, w) = (v, A^*w)\ \forall\ v, w \in V$$

We will consider the bilinear form with arguments Au and $u^\perp$, i.e. $(Au, u^\perp)$, where $u \in U$ and $u^\perp \in U^\perp$.

$$(Au, u^\perp) = 0\ \forall\ u \in U,\ u^\perp \in U^\perp \text{ as } u \text{ and } u^\perp \text{ are orthogonal.}$$

$$(Au, u^\perp) = (u, A^*u^\perp)\ \forall\ u, u^\perp \text{ by definition.}$$

$$\Rightarrow (u, A^*u^\perp) = 0\ \forall\ u \in U,\ u^\perp \in U^\perp$$

$$\Rightarrow u \text{ and } A^*u^\perp \text{ are orthogonal } \forall\ u \in U,\ u^\perp \in U^\perp$$

$$\Rightarrow A^*u^\perp \in U^\perp\ \forall\ u^\perp \in U^\perp$$

$$\therefore\ U^\perp \text{ is } A^*\text{-invariant.}$$

Problem 4

A is an $n \times n$ matrix with n (not necessarily distinct) eigenvalues and corresponding eigenvectors, i.e.

$$Av_i = \lambda_i v_i\ \forall\ 1 \le i \le n,\ \lambda_i \in \mathbb{C},\ v_i \in \mathbb{C}^n.$$

$$A \text{ is a real orthogonal matrix.}$$

$$\Rightarrow \quad ||Av|| = ||v||$$

$$\Rightarrow \quad ||Av_i|| = ||\lambda_i v_i|| = ||v_i||$$

$$\Rightarrow \quad |\lambda_i| \cdot ||v_i|| = ||v_i||$$

$$\Rightarrow \quad |\lambda_i| = 1\ \forall\ 1 \le i \le n$$

$\therefore$ Every eigenvalue of A has absolute value 1.