MAU11102: Linear Algebra II
Homework 8 due 27/03/2020

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Theoretical Physics

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Problem 1

The Gram-Schmidt procedure implies that, given a basis $(v_1, \ldots, v_n)$ of a Euclidean vector space V equipped with a bilinear form (x, y), $(u_1, \ldots, u_n)$ is an orthogonal basis of V, where

$$\begin{aligned} u_{k+1} &= v_{k+1} - \sum_{i=1}^{k} (v_{k+1}, w_i) w_i,\ w_i = \frac{u_i}{||u_i||} \\ &= v_{k+1} - \sum_{i=1}^{k} \left(v_{k+1}, \frac{u_i}{||u_i||} \right) \frac{u_i}{||u_i||} \\ &= v_{k+1} - \sum_{i=1}^{k} \frac{(v_{k+1}, u_i)}{||u_i||^2} u_i,\ u_1 = v_1. \end{aligned}$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ v_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},\ (x, y) = (x \cdot y)$$

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{||u_1||^2} u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$u_2 = v_3 - \frac{v_3 \cdot u_1}{||u_1||^2} u_1 - \frac{v_3 \cdot u_2}{||u_2||^2} u_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{3}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{0}{\frac{2}{3}} \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{Orthogonal basis} = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$$

Problem 2

$$\begin{aligned}(v,u) &= v_1u_1+v_1u_2+v_2u_1+4v_2u_2\\ &= u_1v_1+u_1v_2+u_2v_1+4u_2v_2\\ &= (u,v)\end{aligned}$$

$\Rightarrow$ The bilinear form is symmetric.

$$\begin{aligned}(au+bv,w) &= (au+bv)_1w_1+(au+bv)_1w_2+(au+bv)_2w_1+4(au+bv)_2w_2\\ &= au_1w_1+au_1w_2+au_2w_1+4au_2w_2+bv_1w_1+bv_1w_2+bv_2w_1+4bv_2w_2\\ &= a(u_1w_1+u_1w_2+u_2w_1+4u_2w_2)+b(v_1w_1+v_1w_2+v_2w_1+4v_2w_2)\\ &= a(u,w)+b(v,w)\end{aligned}$$

$\Rightarrow$ The bilinear form is linear in the first argument.

The bilinear form is symmetric. $\Rightarrow$ It must be linear in both arguments.

$$\begin{aligned}(v,v) &= v_1v_1+v_1v_2+v_2v_1+4v_2v_2\\ &= v_1^2+2v_1v_2+4v_2^2\\ &= (v_1+2v_2)^2-v_1(2v_2)\\ &= (a+b)^2-ab,\ a=v_1,\ b=2v_2\end{aligned}$$

$$(a>b \text{ or } a<b) \text{ and } (a,b\geq 0 \text{ or } a,b\leq 0) \Rightarrow (a+b)^2>ab \Rightarrow (v,v)>0$$

$$(a\geq 0,\ b\leq 0,\ a\neq b) \text{ or } (a\leq 0,\ b\geq 0,\ a\neq b) \Rightarrow -ab\geq 0 \Rightarrow (v,v)>0$$

$$a=b\neq 0 \Rightarrow (v,v)=(2a)^2-a^2=3a^2>0$$

$$\Rightarrow (v,v)>0\,\forall\, v\in V\backslash\{0\}$$

$\Rightarrow$ The bilinear form is positive definite.

$\therefore$ The bilinear form is an inner product.

$$w_i=\frac{u_i}{||u_i||},\ u_{k+1}=v_{k+1}-\sum_{i=1}^{k}(v_{k+1},w_i)w_i,\ w_1=\frac{v_1}{||v_1||}$$

$$v_1=\begin{pmatrix}1\\0\end{pmatrix},\ v_2=\begin{pmatrix}0\\1\end{pmatrix},\ (x,y)=x_1y_1+x_1y_2+x_2y_1+4x_2y_2$$

$$w_1=\frac{1}{1}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

$$u_2=v_2-(v_2,w_1)w_1=\begin{pmatrix}0\\1\end{pmatrix}-1\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\Rightarrow w_2=\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$$

$$\Rightarrow \text{Orthonormal basis}=\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}\right)$$

Problem 3

$$
\begin{aligned}
U^\perp &= \{v \in V \mid (v,u) = 0\ \forall\, u \in U\} \\
&= \left\{ v \in \mathbb{R}^4 \;\middle|\; (v \cdot u) = 0\ \forall\, u \in \operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right) \right\} \\
&= \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \;\middle|\; \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0,\ a,b,c,d \in \mathbb{R} \right\}
\end{aligned}
$$

$$
\left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right) r_1 - r_2 \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right) r_2 - r_1 \left(\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right)
$$

$$
\begin{array}{c} a = -d \\ b = -c \end{array} \Rightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ -b \\ -a \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
\Rightarrow U^\perp &= \left\{ a \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \;\middle|\; a,b \in \mathbb{R} \right\} \\
&= \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right)
\end{aligned}
$$

Problem 4

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \ B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}.$$

$$\begin{aligned}
\Rightarrow \sigma(A,B) &= \operatorname{tr}\left(\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}^T \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}\right) \\
&= \operatorname{tr}\left(\begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}\right) \\
&= (a_{11}b_{11} + a_{21}b_{21} + \dots + a_{n1}b_{n1}) + (a_{12}b_{12} + a_{22}b_{22} + \dots + a_{n2}b_{n2}) \\
&\quad + \dots + (a_{1n}b_{1n} + a_{2n}b_{2n} + \dots + a_{nn}b_{nn}) \\
&= \left(\sum_{i=1}^{n} a_{i1}b_{i1}\right) + \left(\sum_{i=1}^{n} a_{i2}b_{i2}\right) + \dots + \left(\sum_{i=1}^{n} a_{in}b_{in}\right) \\
&= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}b_{ij}\right) \\
&= \sum_{i,j=1}^{n} a_{ij}b_{ij}
\end{aligned}$$

$\sigma(B,A)$	$=$	$\sum_{i,j=1}^n b_{ij}a_{ij}$	$\sigma(A,A)$	$=$	$\sum_{i,j=1}^n a_{ij}a_{ij}$
	$=$	$\sum_{i,j=1} a_{ij}b_{ij}$		$=$	$\sum_{i,j=1}^n a_{ij}^2$
	$=$	$\sigma(A,B)$		$>$	$0 \,\forall\, A \in M_{n,n}\backslash\{0\}$
$\Rightarrow$	σ	is symmetric.	$\Rightarrow$	σ	is positive definite.

σ is symmetric, as switching the arguments of σ does not change the expression. σ is positive definite, as when both arguments of σ are the same and $\neq 0$, the expression is always > 0.