MAU11102: Linear Algebra II
Homework 7 due 20/03/2020

Ruaidhrí Campion
19333850
Theoretical Physics

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Problem 1

Let $f(x) = ax^2 + bx + c$ for some $a, b, c \in K$.

(1)

$$f'(x) = 2ax + b \notin K \; \forall \, a \neq 0$$
$$\Rightarrow f \mapsto f' \text{ is not a linear functional.}$$

(2)

$$\begin{aligned} \int_0^1 f(x)dx &= \left. \frac{ax^3}{3} + \frac{bx^2}{2} + cx \right|_0^1 \\ &= \frac{a}{3} + \frac{b}{2} + c \in K \end{aligned}$$
$$\Rightarrow f \mapsto \int_0^1 f(x)dx \text{ is a linear functional.}$$

(3)

$$f'(2) = 2ax + b \,|_2 = 4a + b \in K$$
$$\Rightarrow f \mapsto f'(2) \text{ is a linear functional.}$$

(4)

$$\begin{aligned} \int_0^1 f^2(x)dx &= \int_0^1 (a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2)dx \\ &= \left. \frac{a^2x^5}{5} + \frac{abx^4}{2} + \frac{(2ac + b^2)x^3}{3} + bcx^2 + c^2x \right|_0^1 \\ &= \frac{a^2}{5} + \frac{ab}{2} + \frac{2ac + b^2}{3} + bc + c^2 \in K \end{aligned}$$
$$\Rightarrow f \mapsto \int_0^1 f^2(x)dx \text{ is a linear functional.}$$

Problem 2

Let $e = \left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) = (e_1, e_2, e_3)$ be the given basis of V.

$$e^* = (e_1^*, e_2^*, e_3^*) \text{ is the basis of } V^* \text{ dual to } e$$

$$\Rightarrow e_i^* e_i = 1,\ e_i^* e_j = 0\ \forall\, i \neq j.$$

$$\text{Let } e_1^* = (a_1\ a_2\ a_3),\ e_2^* = (b_1\ b_2\ b_3),\ e_3^* = (c_1\ c_2\ c_3).$$

$$e_1^* e_2 = (a_1\ a_2\ a_3) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = 0 \quad\Rightarrow a_1 = 0$$

$$e_1^* e_1 = (0\ a_2\ a_3) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 \quad\Rightarrow a_3 = -1$$

$$e_1^* e_3 = (0\ a_2\ -1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \quad\Rightarrow a_2 = 1$$

$$e_2^* e_2 = (b_1\ b_2\ b_3) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = 1 \quad\Rightarrow b_1 = -1$$

$$e_2^* e_1 = (-1\ b_2\ b_3) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \quad\Rightarrow b_3 = -1$$

$$e_2^* e_3 = (-1\ b_2\ -1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \quad\Rightarrow b_2 = 1$$

$$e_3^* e_2 = (c_1\ c_2\ c_3) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = 0 \quad\Rightarrow c_1 = 0$$

$$e_3^* e_1 = (0\ c_2\ c_3) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \quad\Rightarrow c_3 = 0$$

$$e_3^* e_3 = (0\ c_2\ 0) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1 \quad\Rightarrow c_2 = 1$$

$$\Rightarrow e^* = ((0\,1\,-1),\ (-1\,1\,-1),\ (0\,1\,0)).$$

Alternatively, we can obtain e^* by finding the inverse of the matrix whose columns are made from e.

$$M = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_3+r_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{r_1-r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\begin{matrix} r_1 + r_2 \\ r_3 - r_2 \end{matrix} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow{r_2 \leftrightarrow -r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$M^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow e^* = ((0\,1\,-1),\ (-1\,1\,-1),\ (0\,1\,0)),\ \text{as before.}$$

Problem 3

Let $e = (1, x, x^2) = (e_1, e_2, e_3)$ be the given basis of V.

$$[\sigma]_e = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},\ a_{ij} = \sigma(e_i, e_j).$$

$$\begin{aligned} \sigma(f,g) &= \int_0^1 (1-x)f(x)g(x)dx \\ &= \int_0^1 (1-x)g(x)f(x)dx \\ &= \sigma(g,f) \Rightarrow \sigma \text{ is symmetric} \Rightarrow a_{ij} = a_{ji}. \end{aligned}$$

$$a_{11} = \sigma(e_1, e_1) = \sigma(1,1) = \int_0^1 (1-x)dx = x - \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$a_{12} = a_{21} = \sigma(e_1, e_2) = \sigma(1, x) = \int_0^1 x(1-x)dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}$$

$$a_{13} = a_{31} = \sigma(e_1, e_3) = \sigma(1, x^2) = \int_0^1 x^2(1-x)dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{12}$$

$$a_{22} = \sigma(e_2, e_2) = \sigma(x, x) = \int_0^1 x^2(1-x)dx = \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{12}$$

$$a_{23} = a_{32} = \sigma(e_2, e_3) = \sigma(x, x^2) = \int_0^1 x^3(1-x)dx = \frac{x^4}{4} - \frac{x^5}{5}\Big|_0^1 = \frac{1}{20}$$

$$a_{33} = \sigma(e_3, e_3) = \sigma(x^2, x^2) = \int_0^1 x^4(1-x)dx = \frac{x^5}{5} - \frac{x^6}{6}\Big|_0^1 = \frac{1}{30}$$

$$\Rightarrow [\sigma]_e = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{6} & \frac{1}{12} & \frac{1}{20} \\ \frac{1}{12} & \frac{1}{20} & \frac{1}{30} \end{pmatrix}$$

Problem 4

$$\text{Let } e = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = (e_1, e_2) \text{ be the standard basis of } \mathbb{R}^2.$$

$$[\sigma]_e = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; a_{ij} = \sigma(e_i, e_j).$$

$$a_{11} = \sigma(e_1, e_1) = \sigma\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = (1)(1) + 2(1)(0) + 3(0)(1) + 4(0)(0) = 1$$

$$a_{12} = \sigma(e_1, e_2) = \sigma\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = (1)(0) + 2(1)(1) + 3(0)(0) + 4(0)(1) = 2$$

$$a_{21} = \sigma(e_2, e_1) = \sigma\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = (0)(1) + 2(0)(0) + 3(1)(1) + 4(1)(0) = 3$$

$$a_{22} = \sigma(e_2, e_2) = \sigma\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = (0)(0) + 2(0)(1) + 3(1)(0) + 4(1)(1) = 4$$

$$\Rightarrow [\sigma]_e = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$[\sigma]_e^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \neq [\sigma]_e \Rightarrow \sigma \text{ is not symmetric.}$$

$$[\sigma]_e^{-1} = \frac{1}{4-6} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$[\sigma]_e$ is invertible $\Rightarrow \sigma$ is non-degenerate.

We can also show that σ is non-degenerate by showing that it is non-degenerate in either argument.

$$\sigma(x, y) = x_1 y_1 + 2x_1 y_2 + 3x_2 y_1 + 4x_2 y_2$$

$$\begin{aligned} \sigma(x, y) = 0 \,\forall\, y \in K^2 \quad &\Rightarrow \quad y_1(x_1 + 3x_2) + y_2(2x_1 + 4x_2) = 0 \,\forall\, y \in K^2 \\ &\Rightarrow \quad x_1 + 3x_2 = 2x_1 + 4x_2 = 0 \,\forall\, y \in K^2 \\ &\Rightarrow \quad x_1 = x_2 = 0 \,\forall\, y \in K^2 \\ &\Rightarrow \quad x = 0 \,\forall\, y \in K^2 \\ &\Rightarrow \quad \sigma \text{ is non-degenerate in the first argument.} \end{aligned}$$

$$\begin{aligned} \sigma(x, y) = 0 \,\forall\, x \in K^2 \quad &\Rightarrow \quad x_1(y_1 + 2y_2) + x_2(3y_1 + 4y_2) = 0 \,\forall\, x \in K^2 \\ &\Rightarrow \quad y_1 + 2y_2 = 3y_1 + 4y_2 = 0 \,\forall\, x \in K^2 \\ &\Rightarrow \quad y_1 = y_2 = 0 \,\forall\, x \in K^2 \\ &\Rightarrow \quad y = 0 \,\forall\, x \in K^2 \\ &\Rightarrow \quad \sigma \text{ is non-degenerate in the second argument.} \end{aligned}$$

$\Rightarrow \sigma$ is non-degenerate.