MAU11102: Linear Algebra II
Homework 6 due 13/03/2020

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Theoretical Physics

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Signed

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Problem 1

(1)

$$t = 0,\ t = 1 \text{ are solutions to the characteristic polynomial}$$

$$\Rightarrow \lambda = 0,\ 1 \text{ are the eigenvalues of } A.$$

$$\lambda = 0 : (A - \lambda I)v = 0$$

$$\left(\begin{array}{cccc|c} 3 & -1 & -2 & 1 & 0 \\ 1 & 0 & -2 & 1 & 0 \\ 5 & -2 & -3 & 2 & 0 \\ 5 & -2 & -4 & 3 & 0 \end{array}\right) \begin{array}{c} r_1 - 3r_2 \\ r_3 - 5r_2 \\ r_4 - 5r_2 \end{array} \left(\begin{array}{cccc|c} 0 & -1 & 4 & -2 & 0 \\ 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & 7 & -3 & 0 \\ 0 & -2 & 6 & -2 & 0 \end{array}\right) \begin{array}{c} r_3 - 2r_1 \\ r_4 - 2r_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 0 & -1 & 4 & -2 & 0 \\ 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{array}\right) \begin{array}{c} r_1 + 4r_3 \\ r_2 - 2r_3 \\ r_4 - 2r_3 \end{array} \left(\begin{array}{cccc|c} 0 & -1 & 0 & 2 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} v_1 = v_4 \\ v_2 = 2v_4 \\ v_3 = v_4 \end{array} \Rightarrow \ker A = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \alpha$$

$$\lambda = 1 : (A - \lambda I)v = 0$$

$$\left(\begin{array}{cccc|c} 2 & -1 & -2 & 1 & 0 \\ 1 & -1 & -2 & 1 & 0 \\ 5 & -2 & -4 & 2 & 0 \\ 5 & -2 & -4 & 2 & 0 \end{array}\right) \begin{array}{c} r_1 - r_2 \\ r_4 - r_3 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -2 & 1 & 0 \\ 5 & -2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) r_2 - r_1$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \\ 5 & -2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) r_3 - 5r_1 - 2r_2 \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} v_1 = 0 \\ v_2 = -2v_3 + v_4 \end{array} \quad \Rightarrow \ker(A-I) = \left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

We need to find generalised eigenvectors to make our Jordan basis,

as the nullity of $A - I$ is too small. Let $B = A - I$.

$$B^2 = \begin{pmatrix} -2 & 1 & 2 & -1 \\ -4 & 2 & 4 & -2 \\ -2 & 1 & 2 & -1 \\ -2 & 1 & 2 & -1 \end{pmatrix}, \ B^3 = \begin{pmatrix} 2 & -1 & -2 & 1 \\ 4 & -2 & -4 & 2 \\ 2 & -1 & -2 & 1 \\ 2 & -1 & -2 & 1 \end{pmatrix} = -B^2$$

$$\Rightarrow \ker(B^2) = \ker(B^3)$$

We can pick a vector in $\ker(B^2)$ that does not exist in $\ker B$ and make a Jordan chain from Av and v. These vectors, along with a vector in $\ker B$, will form Jordan blocks for $\lambda = 1$.

$$B^2 v = 0 : \left(\begin{array}{cccc|c} -2 & 1 & 2 & -1 & 0 \\ -4 & 2 & 4 & -2 & 0 \\ -2 & 1 & 2 & -1 & 0 \\ -2 & 1 & 2 & -1 & 0 \end{array} \right) \begin{array}{c} r_2 - 2r_1 \\ r_3 - r_1 \\ r_4 - r_1 \end{array} \left(\begin{array}{cccc|c} -2 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$2v_1 = v_2 + 2v_3 - v_4 \Rightarrow \ker(B^2) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \right\}$$

$$\text{Let } v = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}. \text{ Then } Bv = \begin{pmatrix} 2 & -1 & -2 & 1 \\ 1 & -1 & -2 & 1 \\ 5 & -2 & -4 & 2 \\ 5 & -2 & -4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{Define some vector in } \ker B \text{ as } \beta, \text{ i.e. } \beta = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

$$\begin{aligned} \text{Then a Jordan basis of } A &= \{\alpha, Bv, v, \beta\} \\ &= \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

(2)

$$\begin{aligned} J &= J_1(0) \oplus J_2(1) \oplus J_1(1) \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

We can also use the fact that $J = M^{-1}AM$, where M is made from the Jordan basis vectors of A.

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & -2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \; M^{-1} = \begin{pmatrix} -2 & 1 & 2 & -1 \\ 2 & -1 & -2 & 2 \\ 3 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow J &= \begin{pmatrix} -2 & 1 & 2 & -1 \\ 2 & -1 & -2 & 2 \\ 3 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -2 & 1 \\ 1 & 0 & -2 & 1 \\ 5 & -2 & -3 & 2 \\ 5 & -2 & -4 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 2 & -1 \\ 2 & -1 & -2 & 2 \\ 3 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 1 & 2 & -1 \\ 2 & -1 & -2 & 2 \\ 3 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ as before.} \end{aligned}$$

Problem 2

(1)

$$\det(tI - A) = \begin{vmatrix} t-1 & -1 \\ 1 & t-3 \end{vmatrix} = (t-1)(t-3) - (-1) = t^2 - 4t + 4 = (t-2)^2$$

$$\Rightarrow \lambda = 2 \text{ is the only eigenvalue of } A.$$

$$(A - 2I)v = 0, \text{ i.e. } \left(\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array} \right)$$

$$v_1 = v_2 \Rightarrow \ker(A - 2I) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A - 2I)^2 v = 0, \text{ i.e. } \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \Rightarrow \ker(A - 2I)^2 = \mathrm{K}^2$$

Pick a vector in $\ker(A - 2I)^2 = \mathrm{K}^2$ not in $\ker(A - 2I)$, i.e. $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$(A - 2I)v = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\text{Then a Jordan basis of } A = \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$J = J_2(2) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

We can also use the fact that $J = M^{-1}AM$, where M is made from the Jordan basis vectors of A.

$$M = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},\ M^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow J &= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \text{ as before.} \end{aligned}$$

(2)

$$f(A) = Mf(J)M^{-1} \Rightarrow A^{100} = MJ^{100}M^{-1}$$

$$J^1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2^1 & 1(2^{1-1}) \\ 0 & 2^1 \end{pmatrix}$$

$$J^2 = \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2^2 & 2(2^{2-1}) \\ 0 & 2^2 \end{pmatrix}$$

$$J^3 = \begin{pmatrix} 8 & 12 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 2^3 & 3(2^{3-1}) \\ 0 & 2^3 \end{pmatrix}$$

$$J^4 = \begin{pmatrix} 16 & 32 \\ 0 & 16 \end{pmatrix} = \begin{pmatrix} 2^4 & 4(2^{4-1}) \\ 0 & 2^4 \end{pmatrix}$$

$$\Rightarrow J^n = \begin{pmatrix} 2^n & n(2^{n-1}) \\ 0 & 2^n \end{pmatrix} = 2^n \begin{pmatrix} 1 & \frac{n}{2} \\ 0 & 1 \end{pmatrix}$$

$$J^{100} = 2^{100} \begin{pmatrix} 1 & 50 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} A^{100} &= \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} 2^{100} \begin{pmatrix} 1 & 50 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \\ &= 2^{100} \begin{pmatrix} -1 & -49 & cc \\ -1 & -50 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \\ &= 2^{100} \begin{pmatrix} -49 & 50 \\ -50 & 51 \end{pmatrix} \end{aligned}$$

Problem 3

$$\lambda = 1 \text{ is an eigenvalue of multiplicity 1.}$$

$$\lambda = 0 \text{ is an eigenvalue of multiplicity 4.}$$

Thus the Jordan normal form must be made up of corresponding Jordan blocks,

i.e. $J = J_1(1) \oplus \oplus_i J_i(0)$, for some i.

Since the rank of A is 2, then the rank of J must also be 2,

i.e. J must have exactly 2 leading entries.

$$\begin{aligned} \text{This is only the case for } J &= J_1(1) \oplus J_2(0) \oplus J_1(0) \oplus J_1(0) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

The largest Jordan blocks associated with each eigenvalue are $J_1(1)$ and $J_2(0)$.

The minimal polynomial of A is a product of some factors of the characteristic polynomial. We can raise each factor to a power corresponding to its largest Jordan block size and get their product to get the minimal polynomial.

$$\Rightarrow p(t) = t^2(t-1)$$