MAU11102: Linear Algebra II
Homework 5 due 12pm, February 28

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Theoretical Physics

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Signed

Date

Problem 1

1.

$$\begin{aligned}
\det(tI - A) &= \begin{vmatrix} t-2 & -1 & 0 \\ -1 & t-3 & -1 \\ 0 & 1 & t-2 \end{vmatrix} \\
&= (t-2)((t-3)(t-2)+1) + (-1(t-2)+0) + 0 \\
&= t^3 - 7t^2 + 16t - 12 = 0
\end{aligned}$$

$$t = 2 \text{ is clearly a solution}$$

$$\frac{t^3 - 7t^2 + 16t - 12}{t-2} = t^2 - 5t + 6$$

The solutions to this quadratic are $t = 2,\ t = 3$

$$\Rightarrow \lambda = 2,\ 3 \text{ are the eigenvalues of } A.$$

2.

$$(A - \lambda I)v = 0$$

$$\lambda = 2 : \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \begin{array}{c} r_2 - r_1 \\ r_3 + r_1 \end{array} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$v_2 = 0, v_1 = -v_3 \Rightarrow v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = 2 \text{ has a multiplicity of } 2$$

$$\Rightarrow \text{ there must be one generalised eigenvector associated with } \lambda = 2$$

$$(A - \lambda I)^2 v = 0$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right)$$

$$v_1 = -v_2 - v_3 \Rightarrow v = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Generalised } v = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = 3 : \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) r_2 + r_1 \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right)$$

$$v_1 = v_2 = -v_3 \Rightarrow v = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$V(2) = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}, V(3) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

3.

$$J = \begin{pmatrix} J(2) & 0 \\ 0 & J(3) \end{pmatrix}$$

$$J(2) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \text{ as there is one eigenvector and}$$

$$\text{one generalised eigenvector associated with } \lambda = 2.$$

$$J(3) = (3), \text{ as there is one eigenvector associated with } \lambda = 3.$$

$$\Rightarrow J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

4.

$$\begin{aligned} \text{Jordan basis} &= \text{span of eigenvectors and generalised eigenvectors} \\ &= \text{span} \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

5.

$$M = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}, \text{ i.e. the matrix made by all eigenvectors of A.}$$

$$\begin{aligned}
M^{-1}AM &= \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 2 & 5 \\ -2 & 0 & -2 \\ -3 & -3 & -3 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \text{ which is a Jordan matrix.}
\end{aligned}$$

Problem 2

1.

$$\begin{aligned}
\det(tI - A) &= \begin{vmatrix} t-2 & 1 & -1 \\ 0 & t-2 & 0 \\ 0 & 1 & t-3 \end{vmatrix} \\
&= (t-2)((t-2)(t-3) - 0(1)) - (0(t-3) - 0(0)) - (0(1) - (t-2)(0)) \\
&= t^3 - 7t^2 + 16t - 12 = 0
\end{aligned}$$

From before, $\lambda = 2,\ 3$ are the eigenvalues of A.

2.

$$(A - \lambda I)v = 0$$

$$\lambda = 2 : \left(\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$v_2 = v_3 \Rightarrow v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = 3 : \left(\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$v_2 = 0, v_1 = v_3 \Rightarrow v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$V(2) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}, V(3) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

3.

$$J = \begin{pmatrix} J(2) & 0 \\ 0 & J(3) \end{pmatrix}$$

$$J(2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},\ J(3) = (3),$$

as each eigenvalue has no generalised eigenvectors associated with it.

$$\Rightarrow J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

4.

$$\text{Jordan basis} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

5.

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},\ \text{i.e. the matrix made by the eigenvectors of A.}$$

$$\begin{aligned} M^{-1}AM &= \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & -2 \\ 0 & 2 & 0 \\ 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},\ \text{which is a Jordan matrix.} \end{aligned}$$