MAU1102: Linear Algebra II Homework 4
due 21/02/2020

Ruaidhrí Campion
19333850

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Problem 1

$$M^{-1}AM = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}^{-1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) r_3 - r_1 \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right) r_2 \leftrightarrow r_3 \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right)$$

$$r_1 + 2r_2 \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 2 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{c} r_1 - r_3 \\ -r_2 \\ -r_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right)$$

$$\begin{aligned} \Rightarrow A &= \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & -3 \\ 0 & 0 & -3 \\ 2 & 0 & -3 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \\ A &= \begin{pmatrix} -2 & 1 & 4 \\ 0 & 3 & 0 \\ -2 & 1 & 4 \end{pmatrix} \end{aligned}$$

Problem 2

$$f(A) = f_n A^n + \cdots + f_1 A + f_0 I = \sum_{i=0}^{n} f_i A^i$$

$$\begin{aligned} \Rightarrow f(A)v &= \sum_{i=0}^{n} f_i A^i v, \text{ where } v \text{ is any arbitrary eigenvector of } A \\ &= \sum_{i=0}^{n} f_i \lambda^i v, \text{ as } Av = \lambda \\ &= f(\lambda) v \end{aligned}$$

$$\begin{aligned} f \text{ annihilates } A &\Rightarrow f(A) = 0 \\ &\Rightarrow f(A)v = f(\lambda)v = 0 \end{aligned}$$

$$v \neq 0 \text{ as } v \text{ is an eigenvector}$$

$$\Rightarrow f(\lambda) = 0$$

Problem 3

(1)

$$
\begin{aligned}
\chi_A(t) &= \det(tI - A) \\
&= \begin{vmatrix} t-3 & -3 & 3 \\ 3 & t+2 & -2 \\ 3 & 2 & t-2 \end{vmatrix} \\
&= (t-3)((t+2)(t-2)+2(2)) + 3(3(t-2)+2(3)) + 3(3(2)-3(t+2)) \\
\chi_A(t) &= t^2(t-3) = t^3 - 3t^2
\end{aligned}
$$

(2)

$$t, t^2 \text{ and } (t-3) \text{ clearly do not annihilate A}$$

$$\Rightarrow p(t) = t(t-3) \text{ or } t^2(t-3)$$

$$
\begin{aligned}
A(A-3I) &= \begin{pmatrix} 3 & 3 & -3 \\ -3 & -2 & 2 \\ -3 & -2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 & -3 \\ -3 & -5 & 2 \\ -3 & -2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{pmatrix} \neq 0
\end{aligned}
$$

$$\Rightarrow p(t) = \chi_A(t) = t^2(t-3) = t^3 - 3t^2$$

(3)

$$\text{The Cayley-Hamilton Theorem states that } \chi_A(A) = 0$$

$$\Rightarrow A^2(A-3I) = 0$$

$$\Rightarrow A^3 = 3A^2$$

$$\text{Multiplying both sides by A gives: } A^4 = 3A^3 = 3(3A^2) = 3^2A^2$$

$$\text{We can repeat this until we have: } A^{2020} = 3^{2018}A^2$$

$$A^2 = \begin{pmatrix} 9 & 9 & -9 \\ -9 & -9 & 9 \\ -9 & -9 & 9 \end{pmatrix} = 3^2 \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow A^{2020} = 3^{2020} \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

Problem 4

$$\begin{aligned}
\det(tI - A) &= \begin{vmatrix} t & -1 & -1 \\ 1 & t-2 & -1 \\ 1 & -1 & t-2 \end{vmatrix} \\
&= t((t-2)^2 - 1) + (t - 2 + 1) - (-1 - t + 2) \\
&= t^3 - 4t^2 + 5t - 2 = 0
\end{aligned}$$

$$t = 1 \text{ is clearly a solution}$$

$$\frac{t^3 - 4t^2 + 5t - 2}{t - 1} = t^2 - 3t + 2$$

$$t = 1,\ t = 2 \text{ are solutions to this quadratic}$$

$$\Rightarrow \lambda = 1,\ \lambda = 1 \text{ and } \lambda = 2 \text{ are the eigenvalues of A.}$$

$$(A - \lambda I)v = 0$$

$$\lambda = 1: \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right) \begin{array}{c} r_2 - r_1 \\ r_3 - r_1 \end{array} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 = x_2 + x_3 \Rightarrow v = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = 2: \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right) \begin{array}{c} r_1 - 2r_2 \\ r_3 - r_2 \end{array} \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) r_3 - r_1 \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 = x_2 = x_3 \Rightarrow v = c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

A is a 3 x 3 matrix and has 3 unique eigenvectors

$\Rightarrow A$ is diagonalisable.

$$\begin{aligned}
\chi_A(t) &= \det(tI - A) \\
&= t^3 - 4t^2 + 5t - 2 \\
&= (t-1)^2(t-2) = 0
\end{aligned}$$

$(t-1)$ and $(t-2)$ clearly don't annihilate A

$$\Rightarrow p(t) = (t-1)(t-2) \text{ or } (t-1)^2(t-2)$$

$$\begin{aligned}
(A - I)(A - 2I) &= \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\Rightarrow p(t) = (t-1)(t-2) = t^2 - 3t + 2$$