MAU1102: Linear Algebra II Homework 3 due 14/02/2020

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Problem 1

If λ is an eigenvalue, then $\det(\lambda I - A) = 0$, i.e.

$$\left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda - 4) - (-1)(2) = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$\lambda = 2, \lambda = 3$ are the eigenvalues of A.

$$Av = \lambda v \Rightarrow (A - \lambda I) v = 0$$

$$\left(\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) v = 0$$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array} \right), \left(\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array} \right)$$

$$r_2 - 2r_1 : \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

For $\lambda = 2,\ v_1 = v_2$. For $\lambda = 3,\ 2v_1 = v_2$.

Thus the eigenvectors of $A = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

We can make a matrix M such that the columns of M are the eigenvectors, i.e. $M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$$\begin{aligned} M^{-1}AM &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \text{ which is a diagonal matrix.} \end{aligned}$$

Problem 2

The subspace $U \subset K^3$ is A-invariant if $Au = w \in U \,\forall\, u \in U$, i.e.

$$\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= \begin{pmatrix} u_2 \\ -4u_1 + 4u_2 \\ -2u_1 + u_2 + 2u_3 \end{pmatrix} \\
&= \begin{pmatrix} c_1 + 3c_2 \\ -4(c_1 + 2c_2) + 4(c_1 + 3c_2) \\ -2(c_1 + 2c_2) + (c_1 + 3c_2) + 2(c_1 + 2c_2) \end{pmatrix} \quad \text{(as } u \text{ is a linear combination of } v_1 \text{ and } v_2\text{)} \\
&= \begin{pmatrix} c_1 + 3c_2 \\ 4c_2 \\ c_1 + 3c_2 \end{pmatrix} \\
&= c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \\
&= c_1 \left(3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right) + c_2 \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right) \\
&= (3c_1 + c_2)v_1 + (-c_1 + c_2)v_2 \\
&= k_1 v_1 + k_2 v_2
\end{aligned}$$

w is also a linear combination of v_1 and v_2, and so $Au = w \in U$, i.e.

$$U \subset K^3 \text{ is } A\text{-invariant.}$$

Problem 3

1.

If λ is an eigenvalue, then $\det(\lambda I - A) = 0$, i.e.

$$\left| \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 5 & 3 & -2 \\ -6 & \lambda + 4 & -4 \\ -4 & 4 & \lambda - 5 \end{vmatrix} = 0$$

$$(\lambda - 5)((\lambda + 4)(\lambda - 5) + 16) - 3((-6\lambda + 30) - 16) - 2(-24 + 4\lambda + 16) = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1 \text{ is clearly a solution}$$

$$\Rightarrow \frac{\lambda^3 - 6\lambda^2 + 11\lambda - 6}{\lambda - 1} = \lambda^2 - 5\lambda + 6 = 0$$

$$\lambda = 2 \text{ and } \lambda = 3 \text{ are solutions to this equation.}$$

Thus the eigenvalues of A are $1, 2$ and 3.

2.

$$Av = \lambda v \Rightarrow (A - Iv) = 0$$

$$\left(\begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right) v = 0$$

$$\left(\begin{array}{ccc|c} 5-\lambda & -3 & 2 & 0 \\ 6 & -4-\lambda & 4 & 0 \\ 4 & -4 & 5-\lambda & 0 \end{array} \right)$$

$$\lambda = 1 : \left(\begin{array}{ccc|c} 4 & -3 & 2 & 0 \\ 6 & -5 & 4 & 0 \\ 4 & -4 & 4 & 0 \end{array} \right) \begin{array}{c} r_2 - \frac{3}{2}r_1 \\ r_3 - r_1 \end{array} \left(\begin{array}{ccc|c} 4 & -3 & 2 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array} \right) \begin{array}{c} r_1 - 6r_2 \\ r_3 - 2r_2 \end{array} \left(\begin{array}{ccc|c} 4 & 0 & -4 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$v_1 = v_3,\ v_2 = 2v_3 \Rightarrow v = c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\lambda = 2 : \left(\begin{array}{ccc|c} 3 & -3 & 2 & 0 \\ 6 & -6 & 4 & 0 \\ 4 & -4 & 3 & 0 \end{array} \right) \begin{array}{c} r_2 - 2r_1 \\ r_3 - \frac{4}{3}r_1 \end{array} \left(\begin{array}{ccc|c} 3 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \end{array} \right) \ r_1 - 6r_3 \ \left(\begin{array}{ccc|c} 3 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \end{array} \right)$$

$$v_1 = v_2,\ v_3 = 0 \Rightarrow v = c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = 3 : \left(\begin{array}{ccc|c} 2 & -3 & 2 & 0 \\ 6 & -7 & 4 & 0 \\ 4 & -4 & 2 & 0 \end{array} \right) \begin{array}{c} r_2 - 3r_1 \\ r_3 - 2r_1 \end{array} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \end{array} \right) \begin{array}{c} r_1 + \frac{3}{2}r_3 \\ r_3 - r_2 \end{array} \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$v_1 = \frac{1}{2}v_3,\ v_2 = v_3 \Rightarrow v = c_3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Thus the eigenvectors of $A = c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, c_3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

3.

We can make a matrix M such that the columns of M are the eigenvectors

$$\text{i.e. } M = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\begin{aligned} M^{-1}AM &= \begin{pmatrix} -2 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 6 \\ 1 & 0 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \text{ which is a diagonal matrix.} \end{aligned}$$