# MAU1102: Linear Algebra II Homework 2 due 07/02/2020

Ruaidhrí Campion
19333850

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## 1 Problem 1

### 1.1

$$\begin{pmatrix} 3 & 2 & 0 & 2 \\ 3 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 \end{pmatrix} \begin{matrix} c_2 - \frac{2}{3}c_1 \\ c_4 - \frac{2}{3}c_1 \end{matrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & -1 & 1 & 1 \\ 2 & \frac{2}{3} & 2 & \frac{2}{3} \\ 2 & \frac{2}{3} & 0 & -\frac{1}{3} \end{pmatrix} \begin{matrix} c_1 + 3c_2 \\ c_3 + c_2 \\ c_4 + c_2 \end{matrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 4 & \frac{2}{3} & \frac{8}{3} & \frac{4}{3} \\ 4 & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$\begin{matrix} c_1 - \frac{3}{2}c_3 \\ c_2 - \frac{1}{4}c_3 \\ c_4 - \frac{1}{2}c_3 \end{matrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 0 \\ 3 & \frac{1}{2} & \frac{2}{3} & 0 \end{pmatrix} \begin{matrix} \frac{1}{3}c_1 \\ -2c_2 \\ \frac{3}{2}c_3 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

$$\text{Basis of } V_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 2 & 2 & 2 \\ 3 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{matrix} c_2 - c_1 \\ c_3 - c_1 \end{matrix} \begin{pmatrix} 2 & 0 & 0 \\ 3 & -2 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{matrix} c_1 + \frac{3}{2}c_2 \\ c_3 - \frac{1}{2}c_2 \end{matrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \\ \frac{5}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

$$c_1 + c_3 \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \\ 2 & 1 & -\frac{1}{2} \end{pmatrix} \begin{matrix} \frac{1}{2}c_1 \\ -c_2 \\ -2c_3 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\text{Basis of } V_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

### 1.2

$$\text{Upon comparison, basis of } V_1 \cap V_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \end{pmatrix} \right\},$$

as these are the only common vectors in the bases of $V_1$ and $V_2$.

### 1.3

Upon comparison, basis of $V_1$ relative to $V_1 \cap V_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}$,

as this is the only vector in the basis of $V_1$ not in the basis of $V_1 \cap V_2$.

## 2 Problem 2

### 2.1

$$(f_1, f_2) \circ [v]_{\mathbf{f}} = v, \text{ i.e. } \left( \begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 1 & 2 \end{array} \right) r_2 - \frac{1}{2} r_1 \left( \begin{array}{cc|c} 2 & 3 & 5 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array} \right) r_1 + 6r_2$$

$$\left( \begin{array}{cc|c} 2 & 0 & 2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array} \right) \begin{array}{c} \frac{1}{2} r_1 \\ -2r_2 \end{array} \left( \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right)$$

$$[v]_{\mathbf{f}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

### 2.2

$$M_{\mathbf{f} \to \mathbf{e}} \circ \mathbf{e} = \mathbf{f}$$

$$M_{\mathbf{f} \to \mathbf{e}} \circ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

$$M_{\mathbf{f} \to \mathbf{e}} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

$$M_{\mathbf{f} \to \mathbf{e}} \circ [v]_{\mathbf{f}} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+3 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = v$$

### 2.3

$$\left( \begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 1 & 0 \end{array} \right) r_2 - \frac{1}{2} r_1 \left( \begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array} \right) r_1 + 6r_2 \left( \begin{array}{cc|c} 2 & 0 & -2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array} \right) \begin{array}{c} \frac{1}{2} r_1 \\ -2r_2 \end{array} \left( \begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array} \right)$$

$$\left( \begin{array}{cc|c} 2 & 3 & 0 \\ 1 & 1 & 1 \end{array} \right) r_2 - \frac{1}{2} r_1 \left( \begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -\frac{1}{2} & 1 \end{array} \right) r_1 + 6r_2 \left( \begin{array}{cc|c} 2 & 0 & 6 \\ 0 & -\frac{1}{2} & 1 \end{array} \right) \begin{array}{c} \frac{1}{2} r_1 \\ -2r_2 \end{array} \left( \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array} \right)$$

$$[e_1]_{\mathbf{f}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, [e_2]_{\mathbf{f}} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

## 2.4

$$M_{\mathbf{e}\to\mathbf{f}} \circ \mathbf{f} = \mathbf{e}$$

$$M_{\mathbf{e}\to\mathbf{f}} \circ \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} M_{\mathbf{e}\to\mathbf{f}} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{2-3} \begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix} \end{aligned}$$

$$M_{\mathbf{e}\to\mathbf{f}} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$$

$$\begin{aligned} M_{\mathbf{e}\to\mathbf{f}} \circ M_{\mathbf{f}\to\mathbf{e}} &= \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} \circ \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2+3 & -3+3 \\ 2-2 & 3-2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$M_{\mathbf{e}\to\mathbf{f}} \circ M_{\mathbf{f}\to\mathbf{e}} = I$$

## 2.5

$$e_1 = -1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = -f_1 + f_2$$

$$e_2 = 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3f_1 - 2f_2$$

$$\begin{aligned} \text{So } \phi(e_1) &= -\phi(f_1) + \phi(f_2) \\ &= -\begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{and } \phi(e_2) &= 3\phi(f_2) - 2\phi(f_2) \\ &= 3\begin{pmatrix} 1 \\ 5 \end{pmatrix} - 2\begin{pmatrix} 3 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ -3 \end{pmatrix} \end{aligned}$$

$$A = [\phi]_{\mathbf{e},\mathbf{e}} \text{ has columns } \phi(e_1) \text{ and } \phi(e_2)$$

$$\Rightarrow A = \begin{pmatrix} 2 & -3 \\ 4 & -3 \end{pmatrix}$$

## 2.6

$$\begin{aligned}
[\phi]_{\mathbf{f},\mathbf{e}} \circ M_{\mathbf{e}\to\mathbf{f}} &= \begin{pmatrix} 1 & 3 \\ 5 & 9 \end{pmatrix} \circ \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} \\
&= \begin{pmatrix} -1+3 & 3-6 \\ -5+9 & 15-18 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -3 \\ 4 & -3 \end{pmatrix} \\
&= A
\end{aligned}$$

This matrix $[\phi]_{\mathbf{f},\mathbf{e}} \circ M_{\mathbf{e}\to\mathbf{f}}$ is the same as $A = [\phi]_{\mathbf{e},\mathbf{e}}$.