MAU11102: Linear Algebra II
Homework 10 due 10/04/2020

Ruaidhrí Campion
19333850
Theoretical Physics

I have read and I understand the plagiarism provisions in the General Regulations of the University Calendar for the current year, found at http://www.tcd.ie/calendar.
I have completed the Online Tutorial in avoiding plagiarism 'Ready, Steady, Write', located at http://tcd-ie.libguides.com/plagiarism/ready-steady-write.

Problem 1

$$\begin{aligned}
q(x) &= x^{\mathbf{T}}Ax \\
&= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} x_1 + x_2 - x_3 \\ x_1 + x_2 \\ -x_1 - x_3 \end{pmatrix} \\
&= x_1^2 + x_2^2 - x_3^2 + 2x_1x_2 - 2x_1x_3 \\
&= x_1^2 + (x_1^2 + 2x_1x_2 + x_2^2) - (x_1^2 + 2x_1x_3 + x_3^2) \\
&= x_1^2 + (x_1 + x_2)^2 - (x_1 + x_3)^2 \\
&= y_1^2 + y_2^2 - y_3^2
\end{aligned}$$

$$x = My \Rightarrow \begin{array}{r} y_1 = x_1 \\ y_2 = x_1 + x_2 \\ y_3 = x_1 + x_3 \end{array} \text{ or } \begin{array}{r} x_1 = y_1 \\ x_2 = -y_1 + y_2 \\ x_3 = -y_1 + y_3 \end{array} \text{ or } M = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{r} \varepsilon_1 = 1 \\ \varepsilon_2 = 1 \\ \varepsilon_3 = -1 \end{array} \Rightarrow \text{Signature of } q = (2, 1, 0)$$

Problem 2

$$q(x) = \sum_{i,j} a_{ij} x_i x_j$$

$$a_{11},\ a_{22},\ a_{33} = 4$$

$$a_{12} + a_{21} = -2,\ a_{13} + a_{31} = 2,\ a_{23} + a_{32} = -2$$

$$A \text{ is symmetric} \Rightarrow a_{ij} = a_{ji}$$

$$\Rightarrow \begin{array}{c} a_{12} = a_{21} = -1 \\ a_{13} = a_{31} = 1 \\ a_{23} = a_{32} = -1 \end{array} \Rightarrow A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}$$

$$\Rightarrow q(x) = x^{\mathbf{T}} \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix} x$$

$$\begin{aligned} \chi_A(t) &= \det(tI - A) \\ &= \begin{vmatrix} t-4 & 1 & -1 \\ 1 & t-4 & 1 \\ -1 & 1 & t-4 \end{vmatrix} \\ &= (t-4)((t-4)(t-4)-1) - (t-4+1) - (1+t-4) \\ &= (t-4)(t^2 - 8t + 15) - 2t + 6 \\ &= t^3 - 12t^2 + 45t + 54 \\ &= (t-3)^2(t-6) \Rightarrow \lambda_1, \lambda_2 = 3,\ \lambda_3 = 6 \end{aligned}$$

$$\begin{aligned} q(x) &= 3y_1^2 + 3y_2^2 + 6y_3^2 \\ &= y^{\mathbf{T}} B y \\ &= y^{\mathbf{T}} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} y \\ &= y^{\mathbf{T}} \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) y \end{aligned}$$

We can find M where $x = My$ such that M is an orthogonal matrix consisting of eigenvectors of A, and $B = M^{\mathbf{T}} AM$. We will find the eigenvectors of A and obtain a set of orthonormal vectors from these, which will make up M.

$$(A - 3I)v = 0 \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) \begin{array}{c} r_2 + r_1 \\ r_3 - r_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$v_1 = v_2 - v_3 \Rightarrow v_{\lambda_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ v_{\lambda_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$(A-6I)v=0 \Rightarrow \left(\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right) \begin{array}{c} r_1+2r_3 \\ r_2+r_3 \end{array} \left(\begin{array}{ccc|c} 0 & -3 & -3 & 0 \\ 0 & -3 & -3 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right)$$

$$\begin{array}{c} -\frac{1}{3}r_1 \\ -\frac{1}{3}r_2 \end{array} \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right) \begin{array}{c} r_1-r_2 \\ r_3+r_2 \end{array} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right)$$

$$\begin{array}{c} v_1 = v_3 \\ v_2 = -v_3 \end{array} \Rightarrow v_{\lambda_3} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$w_1 = \frac{v_{\lambda_1}}{||v_{\lambda_1}||} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$u_2 = v_{\lambda_2} - (v_{\lambda_2}, w_1)w_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$w_2 = \frac{u_2}{||u_2||} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$w_3 = \frac{v_{\lambda_3}}{||v_{\lambda_3}||} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow M = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \middle| \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \middle| \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\begin{array}{rl} & x_1 = \frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \\ \text{or} & x_2 = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 - \frac{1}{\sqrt{3}}y_3 \\ & x_3 = \frac{2}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3 \end{array} \quad \begin{array}{rl} & y_1 = \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 \\ \text{or} & y_2 = -\frac{1}{\sqrt{6}}x_1 + \frac{1}{\sqrt{6}}x_2 + \frac{2}{\sqrt{6}}x_3 \\ & y_3 = \frac{1}{\sqrt{3}}x_1 - \frac{1}{\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3 \end{array}$$

We can already say that q is positive definite, as A is symmetric and all eigenvalues of A are positive $(3, 3, 6)$. We can also say that q is positive definite if all corner minors of A are positive.

$$\begin{aligned} \text{Order 1 corner minor} &= \det(4) \\ &= 4 > 0 \\ \text{Order 2 corner minor} &= \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} \\ &= 4(4) - (-1)(-1) \\ &= 15 > 0 \\ \text{Order 3 corner minor} &= \begin{vmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{vmatrix} \\ &= 4(4(4)-1) + (-4+1) + (1-4) \\ &= 54 > 0 \end{aligned}$$

All corner minors are positive $\Rightarrow q$ is positive definite.

Problem 3

$$\begin{aligned}
q(x) = \sum_{i,j} a_{ij}x_ix_j &= \sum_{i<j}(x_i - x_j)^2,\ 1 \le i,j \le n \\
\Rightarrow \sum_i a_{ii}x_i^2 + \sum_{i\neq j} a_{ij}x_ix_j &= (n-1)\sum_i x_i^2 - 2\sum_{i<j} x_ix_j\ \forall\, x \\
\Rightarrow \sum_i a_{ii}x_i^2 = \sum_i (n-1)x_i^2 &, \quad \sum_{i\neq j} a_{ij}x_ix_j = \sum_{i<j} -2x_ix_j \\
\Rightarrow a_{ii} = n-1\ \forall\, i &, \quad a_{ij} + a_{ji} = -2\ \forall\, i \neq j
\end{aligned}$$

Consider $x_1 = x_2 = \ldots = x_n = 1$, i.e. $x = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. Then $q(x) = 0 \Rightarrow q(x)$ is not positive definite. Thus, A cannot have full rank. If $a_{ij} \neq a_{ji}$ then the columns of A are linearly independent, and thus A has full rank. Thus $a_{ij} = a_{ji} = -1$.

$$\Rightarrow A_n = \begin{pmatrix} n-1 & -1 & \ldots & -1 \\ -1 & n-1 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & n-1 \end{pmatrix}$$

$$\chi_{A_1}(t) = \det(t) = t \Rightarrow \lambda = 0 \Rightarrow (0,0,1)$$

$$\chi_{A_2}(t) = \begin{vmatrix} t-1 & 1 \\ 1 & t-1 \end{vmatrix} = t(t-2) \Rightarrow \lambda = 0,2 \Rightarrow (1,0,1)$$

$$\chi_{A_3}(t) = \begin{vmatrix} t-2 & 1 & 1 \\ 1 & t-2 & 1 \\ 1 & 1 & t-2 \end{vmatrix} = t(t-3)^2 \Rightarrow \lambda = 0,3,3 \Rightarrow (2,0,1)$$

$$\chi_{A_4}(t) = \begin{vmatrix} t-3 & 1 & 1 & 1 \\ 1 & t-3 & 1 & 1 \\ 1 & 1 & t-3 & 1 \\ 1 & 1 & 1 & t-3 \end{vmatrix} = t(t-4)^3 \Rightarrow \lambda = 0,4,4,4 \Rightarrow (3,0,1)$$

$$\chi_{A_5}(t) = \begin{vmatrix} t-4 & 1 & 1 & 1 & 1 \\ 1 & t-4 & 1 & 1 & 1 \\ 1 & 1 & t-4 & 1 & 1 \\ 1 & 1 & 1 & t-4 & 1 \\ 1 & 1 & 1 & 1 & t-4 \end{vmatrix} = t(t-5)^4 \Rightarrow \lambda = 0,5,5,5,5 \Rightarrow (4,0,1)$$

$$\vdots$$

$$\Rightarrow \chi_{A_n}(t) = t(t-n)^{n-1} \Rightarrow \lambda = 0,n,n,\ldots,n \Rightarrow (n-1,0,1)$$

$\therefore$ Signature of $q(x) = (n-1,0,1)$

Alternatively, we can use the fact that A is symmetric to find n_+, n_- and n_0. There exists only one vector in $\ker(A) \Rightarrow n_0 = 1$ and $\text{rk}(A) = n_+ + n_- = n-1$. $q(x) \geq 0 \Rightarrow n_- = 0 \Rightarrow n_+ = n-1$.

$\therefore$ Signature of $q(x) = (n-1, 0, 1)$, as before.

Problem 4

$$
\begin{aligned}
q(x) &= x^{\mathbf{T}} A x \\
&= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= x_1^2 + a x_2^2 + 4 x_1 x_2 \\
&= (x_1^2 + 4 x_1 x_2 + 4 x_2^2) + (a-4) x_2^2 \\
&= (x_1 + 2x_2)^2 + (a-4) x_2^2 \\
&> 0 \ \forall \ x_1, x_2 \in \mathbb{R} \backslash \{0\} \\
(x_1 + 2x_2)^2 &\geq 0 \\
\Rightarrow (a-4) x_2^2 &> 0 \\
\\
\Rightarrow a &> 4
\end{aligned}
$$