MAU11202: Advanced Calculus
Homework 3 due 27/03/2020

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Theoretical Physics

1.

$$f(x,y) = (x-2)^2 + (y-1)^2$$
$$g(x,y) = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 - 1$$
$$\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$$
$$\Rightarrow 2(x-2)\hat{\imath} + 2(y-1)\hat{\jmath} = \frac{\lambda x}{8}\hat{\imath} + \frac{2\lambda y}{9}\hat{\jmath}$$
$$\Rightarrow x = \frac{32}{16-\lambda},\ y = \frac{9}{9-\lambda}$$
$$g(x,y) = 0$$
$$\Rightarrow \left(\frac{8}{16-\lambda}\right)^2 + \left(\frac{3}{9-\lambda}\right)^2 = 1$$
$$64(9-\lambda)^2 + 9(16-\lambda)^2 = ((16-\lambda)(9-\lambda))^2$$
$$64(81-18\lambda+\lambda^2) + 9(256-32\lambda+\lambda^2) = (144-25\lambda+\lambda^2)^2$$
$$7488 - 1440\lambda + 73\lambda^2 = 20736 - 7200\lambda + 288\lambda^2 + 625\lambda^2 - 50\lambda^3 + \lambda^4$$
$$\lambda^4 - 50\lambda^3 + 840\lambda^2 - 5760\lambda + 13248 = 0$$

We cannot solve this analytically, and so we will use the Newton-Raphson iteration.

$$\lambda_{n+1} = \lambda_n - \frac{h(\lambda_n)}{h'(\lambda_n)}$$
$$h(\lambda) = \lambda^4 - 50\lambda^3 + 840\lambda^2 - 5760\lambda + 13248$$
$$h'(\lambda) = 4\lambda^3 - 150\lambda^2 + 1680\lambda - 5760$$

Graphing this function gives approximate roots $\lambda = 5,\ \lambda = 24$

$\lambda_0 = 5$	$\lambda_0 = 24$
$\lambda_1 = \frac{2873}{610}$	$\lambda_1 = \frac{145}{6}$
$\lambda_2 = 4.737508589$	$\lambda_2 = 24.16137202$
$\lambda_3 = 4.737782761$	$\lambda_3 = 24.16136655$
$\lambda_4 = 4.737782788$	$\lambda_4 = 24.16136655 \equiv \lambda_b$
$\lambda_5 = 4.737782788 \equiv \lambda_a$	

$$f_a(x,y) = \left(\frac{32}{16-\lambda_a} - 2\right)^2 + \left(\frac{9}{9-\lambda_a} - 1\right)^2 = 1.943488147 = f_{min}$$
$$f_b(x,y) = \left(\frac{32}{16-\lambda_b} - 2\right)^2 + \left(\frac{9}{9-\lambda_b} - 1\right)^2 = 37.59680455 = f_{max}$$

2.

$$\begin{aligned}
\int\int_R f(x,y)dA &= \int_0^3 \int_0^2 4xye^{x^2+y^2}dxdy \\
&= \int_0^3 2y \int_0^2 2xe^{x^2+y^2}dxdy \\
&= \int_0^3 2ye^{x^2+y^2}\Big|_{x=0}^{x=2} dy \\
&= e^{x^2+y^2}\Big|_{x=0}^{x=2}\Big|_{y=0}^{y=3} \\
&= e^{y^2+4} - e^{y^2}\Big|_{y=0}^{y-3} \\
&= e^{13} - e^9 - e^4 + 1 = (e^9-1)(e^4-1) = 434256.7099
\end{aligned}$$

3.

(a)

$$\begin{aligned}
x &= 2 \pm \sqrt{7+12y-4y^2} \\
&= 2 \pm \sqrt{(2y-7)(2y+1)},\ -\frac{1}{2} \le y \le \frac{7}{2}
\end{aligned}$$

$$\frac{dx}{dy} = \pm\frac{12-8y}{2\sqrt{7+12y-4y^2}}$$

$$\frac{dx}{dy} = 0 \Rightarrow y = \frac{3}{2} \Rightarrow x = 2 \pm 4$$

$\Rightarrow$ Ellipse, where $-2 \le x \le 6,\ -\frac{1}{2} \le y \le \frac{7}{2}$ (sketch on page 3)

$$x = 2 \pm \sqrt{7+12y-4y^2}$$

$$(x-2)^2 = 7+12y-4y^2$$

$$4y^2 - 12y + ((x-2)^2 - 7) = 0$$

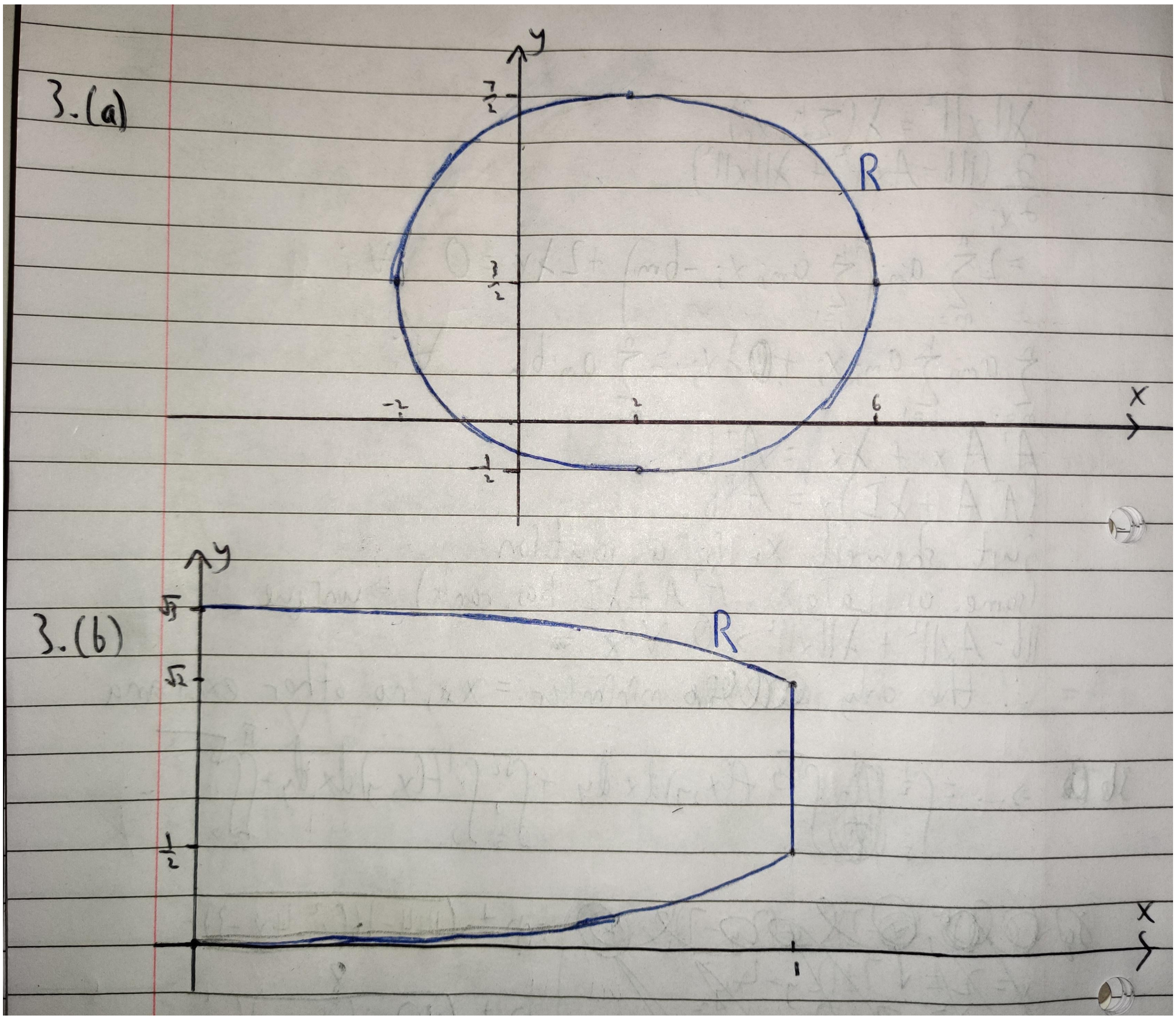
$$\begin{aligned}
y &= \frac{12 \pm \sqrt{144 - 16((x-2)^2-7)}}{8} \\
&= \frac{12 \pm \sqrt{144-16(x^2-4x-3)}}{8} \\
&= \frac{12 \pm 4\sqrt{12+4x-x^2}}{8} \\
&= \frac{3 \pm \sqrt{12+4x-x^2}}{2},\ -2 \le x \le 6
\end{aligned}$$

$$\Rightarrow \int_{-2}^{6} \int_{\frac{3-\sqrt{12+4x-x^2}}{2}}^{\frac{3+\sqrt{12+4x-x^2}}{2}} f(x,y)dydx$$

(b)

$$\frac{x^2}{2} \leq y \leq \sqrt{3-x^2},\ 0 \leq x \leq 1$$
$$\Rightarrow \sqrt{2y} \leq x \leq \sqrt{3-y^2}$$
$$0 \leq \sqrt{2y} \leq 1 \Rightarrow 0 \leq y \leq \frac{1}{2}$$
$$0 \leq \sqrt{3-y^2} \leq 1 \Rightarrow \sqrt{2} \leq y \leq \sqrt{3}$$
$$\Rightarrow \text{Graphs don't intersect, need } x = 1 \text{ for } \frac{1}{2} \leq y \leq \sqrt{2} \text{ (sketch below)}$$

$$\Rightarrow \int_0^{\frac{1}{2}} \int_0^{\sqrt{2y}} f(x,y)dxdy + \int_{\frac{1}{2}}^{\sqrt{2}} \int_0^{1} f(x,y)dxdy + \int_{\sqrt{2}}^{\sqrt{3}} \int_0^{\sqrt{3-y^2}} f(x,y)dxdy$$



4.

(a)

$$f(x,y) = x^2 + y^2,\ y^2 \le x \le 1,\ -1 \le y \le 1$$

$$\begin{aligned}
\Rightarrow V &= \int_{-1}^{1}\int_{y^2}^{1}(x^2+y^2)dxdy \\
&= \int_{-1}^{1}\left(\frac{x^3}{3}+xy^2\right)\Bigg|_{x=y^2}^{x=1} dy \\
&= \int_{-1}^{1}\left(\frac{1}{3}+y^2-\frac{y^6}{3}-y^4\right)dy \\
&= \frac{y}{3}+\frac{y^3}{3}-\frac{y^7}{21}-\frac{y^5}{5}\Bigg|_{y=-1}^{y=1} \\
&= \frac{88}{105}
\end{aligned}$$

(b)

$$f(x,y) = \frac{x^2}{a},\ 0 \le y \le \sqrt{b^2-x^2},\ 0 \le x \le b$$

$$\Rightarrow V = \int_0^b \int_0^{\sqrt{b^2-x^2}} \frac{x^2}{a^2} dydx$$

$$x = r\cos\theta,\ y = r\sin\theta$$

$$\Rightarrow 0 \le r \le b,\ 0 \le \theta \le \frac{\pi}{2}$$

$$\begin{aligned}
\Rightarrow V &= \int_0^{\frac{\pi}{2}}\int_0^b \frac{(r\cos\theta)^2}{a} rdrd\theta \\
&= \frac{1}{a}\int_0^{\frac{\pi}{2}} \cos^2\theta \int_0^b r^3 drd\theta \\
&= \frac{1}{a}\int_0^{\frac{\pi}{2}} \frac{r^4\cos^2\theta}{4}\Bigg|_{r=0}^{r=b} d\theta \\
&= \frac{1}{a}\int_0^{\frac{\pi}{2}} \frac{b^4\cos^2\theta}{4} d\theta \\
&= \frac{b^4}{4a}\left(\frac{1}{2}\left(\theta+\frac{\sin(2\theta)}{2}\right)\right)\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}} \\
&= \frac{b^4}{8a}\left(\frac{\pi}{2}\right) \\
&= \frac{\pi b^4}{16a}
\end{aligned}$$

5.

$$f(x,y,z) = s^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$
$$g(x,y,z) = ax + by + cz + d$$
$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0)$$
$$\Rightarrow 2(x-x_0)\hat{\imath} + 2(y-y_0)\hat{\jmath} + 2(z-z_0)\hat{k} = \lambda a\hat{\imath} + \lambda b\hat{\jmath} + \lambda c\hat{k}$$
$$\Rightarrow x = x_0 + \frac{\lambda a}{2},\ y = y_0 + \frac{\lambda b}{2},\ z = z_0 + \frac{\lambda c}{2}$$
$$g(x,y,z) = 0$$
$$\Rightarrow a\left(x_0 + \frac{\lambda a}{2}\right) + b\left(y_0 + \frac{\lambda b}{2}\right) + c\left(z_0 + \frac{\lambda c}{2}\right) + d = 0$$
$$\frac{\lambda}{2}(a^2+b^2+c^2) = -(ax_0 + by_0 + cz_0 + d)$$
$$\frac{\lambda}{2} = -\frac{ax_0 + by_0 + cz_0 + d}{a^2+b^2+c^2}$$

$$\begin{aligned}
s^2 &= (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \\
&= \left(\frac{\lambda}{2}a\right)^2 + \left(\frac{\lambda}{2}b\right)^2 + \left(\frac{\lambda}{2}c\right)^2 \\
&= \left(\frac{\lambda}{2}\right)^2 (a^2+b^2+c^2) \\
&= \left(-\frac{ax_0 + by_0 + cz_0 + d}{a^2+b^2+c^2}\right)^2 (a^2+b^2+c^2) \\
&= \frac{(ax_0 + by_0 + cz_0 + d)^2}{a^2+b^2+c^2}
\end{aligned}$$

$$\Rightarrow s = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

6.

$$f(x_1, x_2, \ldots, x_n) = (2x_1)(2x_2)\ldots(2x_n) = 2^n x_1 x_2 \ldots x_n$$

$$g(x_1, x_2, \ldots, x_n) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \ldots + \frac{x_n^2}{a_n^2} - 1$$

$$\nabla f(x_1, x_2, \ldots, x_n) = \lambda \nabla g(x_1, x_2, \ldots, x_n)$$

$$2^n x_2 x_3 \ldots x_n \hat{e_1} + 2^n x_1 x_3 \ldots x_n \hat{e_2} + \ldots + 2^n x_1 x_2 \ldots x_{n-1} \hat{e_n} = \frac{2\lambda x_1}{a_1^2}\hat{e_1} + \frac{2\lambda x_2}{a_2^2}\hat{e_2} + \ldots + \frac{2\lambda x_n}{a_n^2}\hat{e_n}$$

$$\Rightarrow \frac{2^n x_1 x_2 \ldots x_n}{x_i} = \frac{2\lambda x_i}{a_i^2}$$

$$\Rightarrow \frac{x_i^2}{a_i^2} = \frac{f(x_1, x_2, \ldots, x_n)}{2\lambda}$$

$$g(x_1, x_2, \ldots, x_n) = 0$$

$$\Rightarrow \frac{f(x_1, x_2, \ldots, x_n)}{2\lambda} + \frac{f(x_1, x_2, \ldots, x_n)}{2\lambda} + \ldots + \frac{f(x_1, x_2, \ldots, x_n)}{2\lambda} = 1$$

$$\frac{nf(x_1, x_2, \ldots, x_n)}{2\lambda} = 1 \Rightarrow 2\lambda = nf(x_1, x_2, \ldots, x_n)$$

$$\Rightarrow \frac{x_i^2}{a_i^2} = \frac{1}{n} \Rightarrow x_i = \frac{a_i}{\sqrt{n}}$$

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= 2^n x_1 x_2 \ldots x_n \\ &= 2^n \left(\frac{a_1}{\sqrt{n}}\right)\left(\frac{a_2}{\sqrt{n}}\right)\ldots\left(\frac{a_n}{\sqrt{n}}\right) \\ &= \left(\frac{2}{\sqrt{n}}\right)^n a_1 a_2 \ldots a_n = \left(\frac{2}{\sqrt{n}}\right)^n \prod_{i=1}^{n} a_i \end{aligned}$$

7.

$$\mathbf{u} = [x_1\ x_2\ \ldots\ x_n]^T,\ \mathbf{v} = [1\ 2\ \ldots\ n]^T$$

$$||\mathbf{u}|| = 1 \Rightarrow \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = 1 \Rightarrow x_1^2 + x_2^2 + \ldots + x_n^2 - 1 = 0$$

$$f(x_1, x_2, \ldots, x_n) = \mathbf{u} \cdot \mathbf{v} = x_1 + 2x_2 + \ldots + nx_n$$

$$g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 - 1$$

$$\nabla f(x_1, x_2, \ldots, x_n) = \lambda \nabla g(x_1, x_2, \ldots, x_n)$$

$$\Rightarrow \hat{e_1} + 2\hat{e_2} + \ldots + n\hat{e_n} = \lambda(2x_1\hat{e_1} + 2x_2\hat{e_2} + \ldots + 2x_n\hat{e_n})$$

$$\Rightarrow x_1 = \frac{1}{2\lambda},\ x_2 = \frac{2}{2\lambda},\ \ldots,\ x_n = \frac{n}{2\lambda}$$

$$g(x_1, x_2, \ldots, x_n) = 0$$

$$\Rightarrow \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{2}{2\lambda}\right)^2 + \ldots + \left(\frac{n}{2\lambda}\right)^2 = 1$$

$$2\lambda = \sqrt{1^2 + 2^2 + \ldots + n^2} = ||\mathbf{v}||$$

$$\begin{aligned} \mathbf{u} &= [x_1\ x_2\ \ldots\ x_n]^T \\ &= \left[\frac{1}{2\lambda}\ \frac{2}{2\lambda}\ \ldots\ \frac{n}{n\lambda}\right]^T \\ &= \frac{1}{2\lambda}[1\ 2\ \ldots\ n]^T \\ &= \frac{1}{||\mathbf{v}||}\mathbf{v} \end{aligned}$$

8.

$$f(x_1, x_2, \ldots, x_n) = (2x_1)(2x_2)\ldots(2x_n) = 2^n x_1 x_2 \ldots x_n$$

$$g(x_1, x_2, \ldots, x_n) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \ldots + \frac{x_n^2}{a_n^2} - 1$$

$$\nabla f(x_1, x_2, \ldots, x_n) = \lambda \nabla g(x_1, x_2, \ldots, x_n)$$

$$2^n x_2 x_3 \ldots x_n \hat{e_1} + 2^n x_1 x_3 \ldots x_n \hat{e_2} + \ldots + 2^n x_1 x_2 \ldots x_{n-1} \hat{e_n} = \frac{2\lambda x_1}{a_1^2}\hat{e_1} + \frac{2\lambda x_2}{a_2^2}\hat{e_2} + \ldots + \frac{2\lambda x_n}{a_n^2}\hat{e_n}$$

$$\Rightarrow \frac{2^n x_1 x_2 \ldots x_n}{x_i} = \frac{2\lambda x_i}{a_i^2}$$

$$\Rightarrow \frac{x_i^2}{a_i^2} = \frac{f(x_1, x_2, \ldots, x_n)}{2\lambda}$$

$$g(x_1, x_2, \ldots, x_n) = 0$$

$$\Rightarrow \frac{f(x_1, x_2, \ldots, x_n)}{2\lambda} + \frac{f(x_1, x_2, \ldots, x_n)}{2\lambda} + \ldots + \frac{f(x_1, x_2, \ldots, x_n)}{2\lambda} = 1$$

$$\frac{nf(x_1, x_2, \ldots, x_n)}{2\lambda} = 1 \Rightarrow 2\lambda = nf(x_1, x_2, \ldots, x_n)$$

$$\Rightarrow \frac{x_i^2}{a_i^2} = \frac{1}{n} \Rightarrow x_i = \frac{a_i}{\sqrt{n}}$$

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= 2^n x_1 x_2 \ldots x_n \\ &= 2^n \left(\frac{a_1}{\sqrt{n}}\right)\left(\frac{a_2}{\sqrt{n}}\right)\ldots\left(\frac{a_n}{\sqrt{n}}\right) \\ &= \left(\frac{2}{\sqrt{n}}\right)^n a_1 a_2 \ldots a_n = \left(\frac{2}{\sqrt{n}}\right)^n \prod_{i=1}^{n} a_i \end{aligned}$$

9.

(a)

$$
\begin{aligned}
\mathbf{b} - \mathbf{A}\mathbf{x} &= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \\
&= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} - \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1k}x_k \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2k}x_k \\ \vdots a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nk}x_k \end{pmatrix} \\
&= \begin{pmatrix} b_1 - a_{11}x_1 - a_{12}x_2 - \dots - a_{1k}x_k \\ b_2 - a_{21}x_1 - a_{22}x_2 - \dots - a_{2k}x_k \\ \vdots \\ b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nk}x_k \end{pmatrix}
\end{aligned}
$$

$$||\mathbf{b}-\mathbf{A}\mathbf{x}||^2 = (b_1-a_{11}x_1-a_{12}x_2-\dots-a_{1k}x_k)^2+\dots+(b_n-a_{n1}x_1-a_{n2}x_2-\dots-a_{nk}x_k)^2$$

$$
\begin{aligned}
\frac{\partial}{\partial x_i}||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 &= 2(b_1 - a_{11}x_1 - a_{12}x_2 - \dots - a_{1k}x_k)(-a_{1i}) + \dots \\
&+ 2(b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nk}x_k)(-a_{ni}) \\
&= -2a_{1i}\left(b_1 - \sum_{j=1}^{k} a_{1j}x_j\right) - \dots - 2a_{ni}\left(b_n - \sum_{j=1}^{k} a_{nj}x_j\right) \\
&= 2\sum_{m=1}^{n} a_{mi}\left(\sum_{j=1}^{k} a_{mj}x_j - b_m\right)
\end{aligned}
$$

(b)

$$\begin{aligned}
\mathbf{A}^T\mathbf{A}\mathbf{x}_{min} &= \mathbf{A}^T \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nk} \end{pmatrix} \begin{pmatrix} x_{min,1} \\ x_{min,2} \\ \vdots \\ x_{min,k} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nk} \end{pmatrix}^T \begin{pmatrix} a_{11}x_{min,1} + a_{12}x_{min,2} + \ldots + a_{1k}x_{min,k} \\ a_{21}x_{min,1} + a_{22}x_{min,2} + \ldots + a_{2k}x_{min,k} \\ \vdots \\ a_{n1}x_{min,1} + a_{n2}x_{min,2} + \ldots + a_{nk}x_{min,k} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1k} & a_{2k} & \ldots & a_{nk} \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{k} a_{1j}x_{min,j} \\ \sum_{j=1}^{k} a_{2j}x_{min,j} \\ \vdots \\ \sum_{j=1}^{k} a_{nj}x_{min,j} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}\sum_{j=1}^{k} a_{1j}x_{min,j} + a_{21}\sum_{j=1}^{k} a_{2j}x_{min,j} + \ldots + a_{n1}\sum_{j=1}^{k} a_{nj}x_{min,j} \\ a_{12}\sum_{j=1}^{k} a_{1j}x_{min,j} + a_{22}\sum_{j=1}^{k} a_{2j}x_{min,j} + \ldots + a_{n2}\sum_{j=1}^{k} a_{nj}x_{min,j} \\ \vdots \\ a_{1k}\sum_{j=1}^{k} a_{1j}x_{min,j} + a_{2k}\sum_{j=1}^{k} a_{2j}x_{min,j} + \ldots + a_{nk}\sum_{j=1}^{k} a_{nj}x_{min,j} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{m=1}^{n} a_{m1}\sum_{j=1}^{k} a_{mj}x_{min,j} \\ \sum_{m=1}^{n} a_{m2}\sum_{j=1}^{k} a_{mj}x_{min,j} \\ \vdots \\ \sum_{m=1}^{n} a_{mk}\sum_{j=1}^{k} a_{mj}x_{min,j} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^T\mathbf{b} &= \begin{pmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1k} & a_{2k} & \ldots & a_{nk} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\
&= \begin{pmatrix} a_{11}b_1 + a_{21}b_2 + \ldots + a_{n1}b_n \\ a_{12}b_1 + a_{22}b_2 + \ldots + a_{n2}b_n \\ \vdots \\ a_{1k}b_1 + a_{2k}b_2 + \ldots + a_{nk}b_n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{m=1}^{n} a_{m1}b_m \\ \sum_{m=1}^{n} a_{m2}b_m \\ \vdots \\ \sum_{m=1}^{n} a_{mk}b_m \end{pmatrix}
\end{aligned}$$

$$\mathbf{x}_{min} \text{ minimises } ||\mathbf{b} - \mathbf{A}\mathbf{x}||$$

$$\Rightarrow \mathbf{x}_{min} \text{ minimises } ||\mathbf{b} - \mathbf{A}\mathbf{x}||^2$$

$$\Rightarrow \frac{\partial}{\partial x_{min,i}} ||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 = 0 \;\forall\, 1 \le i \le k$$

$$\Rightarrow 2\sum_{m=1}^{n} a_{mi}\left(\sum_{j=1}^{k} a_{mj}x_{min,j} - b_m\right) = 0 \,\forall\, 1 \leq i \leq k$$

$$\Rightarrow \sum_{m=1}^{n} a_{mi}\sum_{j=1}^{k} a_{mj}x_{min,j} = \sum_{m=1}^{n} a_{mi}b_m \,\forall\, 1 \leq i \leq k$$

$$\Rightarrow i^{\text{th}} \text{ row of } \mathbf{A}^T\mathbf{A}\mathbf{x}_{min} = i^{\text{th}} \text{ row of } \mathbf{A}^T\mathbf{b} \,\forall\, 1 \leq i \leq k$$

$$\therefore \mathbf{A}^T\mathbf{A}\mathbf{x}_{min} = \mathbf{A}^T\mathbf{b}$$

(c)

$\mathbf{A}^T\mathbf{A}\mathbf{x}_0 = \mathbf{A}^T\mathbf{b}$ holds if $\mathbf{x}_0$ is a local maximiser or minimiser of $r(\mathbf{x})$.

$\mathbf{A}^T\mathbf{b}$ has no dependence on $\mathbf{x}$ and is thus constant.

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T) = \text{rank}(\mathbf{A}^T\mathbf{A}) = k$$

$\Rightarrow \mathbf{A}^T\mathbf{A}$ is a $k \times k$ matrix of rank k.

$\Rightarrow$ There exists a unique $\mathbf{x}_0$ such that $\mathbf{A}^T\mathbf{A}\mathbf{x}_0 = \mathbf{A}^T\mathbf{b}$

$\Rightarrow r(\mathbf{x})$ has one global extremum.

$$r(\mathbf{x}) = ||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 \geq 0 \,\forall\, \mathbf{x}$$

$\Rightarrow$ If $r(\mathbf{x})$ has only one global extremum, it must be a global minimum.

$\therefore r(\mathbf{x})$ has a global minimiser and no other extrema.

(d)

$$f(\mathbf{x}) = ||\mathbf{b} - \mathbf{A}\mathbf{x}||^2$$

$$g(\mathbf{x}) = ||\mathbf{x}|| - R = \sqrt{\sum_{p=1}^{k} x_p^2} - R$$

$$\nabla f(\mathbf{x}) = \delta \nabla g(\mathbf{x}),\ \delta \in \mathbb{R}$$

$$\Rightarrow \frac{\partial}{\partial x_i}||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 \hat{e}_i = \delta \frac{\partial}{\partial x_i}(||\mathbf{x}|| - R)\hat{e}_i \,\forall\, 1 \leq i \leq k$$

$$2\sum_{m=1}^{n} a_{mi}\left(\sum_{j=1}^{k} a_{mj}x_j - b_m\right) = \delta\left(\frac{1}{2}\left(\sum_{p=1}^{k} x_p^2\right)^{-\frac{1}{2}} 2x_i\right)$$

$$\text{Let } \varepsilon_i = 2\sum_{m=1}^{n} a_{mi}\left(\sum_{j=1}^{k} a_{mj}x_j - b_m\right)$$

$$\Rightarrow \varepsilon_i = \frac{\delta}{\sqrt{\sum_{p=1}^{k} x_p^2}} x_i = \frac{\delta}{||\mathbf{x}||} x_i$$

$$g(\mathbf{x}) = 0 \Rightarrow ||\mathbf{x}|| = R \Rightarrow \varepsilon_i = \frac{\delta}{R} x_i$$

$$\Rightarrow 2 \sum_{m=1}^{n} a_{mi} \left(\sum_{j=1}^{k} a_{mj} x_j - b_m \right) = \frac{\delta}{R} x_i$$

$$2 \sum_{m=1}^{n} a_{mi} \sum_{j=1}^{k} a_{mj} x_j - \frac{\delta}{R} x_i = 2 \sum_{m=1}^{n} a_{mi} b_m \; \forall \, 1 \leq i \leq k$$

$$\Rightarrow \left(2\mathbf{A}^T \mathbf{A} - \frac{\delta}{R} \mathbf{I} \right) \mathbf{x} = 2\mathbf{A}^T \mathbf{b}$$

$$\text{Let } \delta = -2R\lambda \Rightarrow \mathbf{x} = \mathbf{x}_\lambda$$

$$\Rightarrow (\mathbf{A}^T \mathbf{A} + \lambda \mathbf{I}) \mathbf{x}_\lambda = \mathbf{A}^T \mathbf{b}$$

$$\lambda ||\mathbf{x}||^2 = \lambda \left(\sum_{p=1}^{k} x_p^2 \right)$$

$$\frac{\partial}{\partial x_i} (||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 + \lambda ||\mathbf{x}||^2) = 0 \; \forall \, 1 \leq i \leq k$$

$$2 \sum_{m=1}^{n} a_{mi} \left(\sum_{j=1}^{k} a_{mj} x_j - b_m \right) + 2\lambda x_i = 0$$

$$\sum_{m=1}^{n} a_{mi} \sum_{j=1}^{k} a_{mj} x_j + \lambda x_i = \sum_{m=1}^{n} a_{mi} b_m \; \forall \, 1 \leq i \leq k$$

$$\Rightarrow \mathbf{A}^T \mathbf{A}\mathbf{x} + \lambda \mathbf{x} = \mathbf{A}^T \mathbf{b}$$

$$\Rightarrow (\mathbf{A}^T \mathbf{A} + \lambda \mathbf{I}) \mathbf{x} = \mathbf{A}^T \mathbf{b}$$

We just showed that $\mathbf{x}_\lambda$ is a solution of this linear system.

$\Rightarrow \mathbf{x}_\lambda$ is a maximiser or minimiser of $||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 + \lambda ||\mathbf{x}||^2$

$\mathbf{A}^T \mathbf{b}$ has no dependence on $\mathbf{x}$ and is thus constant.

$\mathbf{A}^T \mathbf{A} + \lambda \mathbf{I}$ is a $k \times k$ matrix of rank k.

$\Rightarrow \mathbf{x}_\lambda$ is a unique solution of this linear system.

$\Rightarrow ||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 + \lambda ||\mathbf{x}||^2$ has one global extremum at $\mathbf{x}_\lambda$.

$$||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 + \lambda ||\mathbf{x}||^2 \geq 0 \; \forall \, \mathbf{x}$$

$\Rightarrow$ If $||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 + \lambda ||\mathbf{x}||^2$ has one global extremum, it must be a global minimum.

$\therefore \mathbf{x}_\lambda$ is the only minimiser of $||\mathbf{b} - \mathbf{A}\mathbf{x}||^2 + \lambda ||\mathbf{x}||^2$