Quantum Mechanics

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A lengthier approach

Electromagnetic Waves 2.1
Maxwell in 1864 proposed that a changing electric field has a magnetic field associated with it

\[ c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.998 \times 10^8 \text{ms}^{-1} \]

\( \varepsilon_0 = \) electric permittivity of free space
\( \mu_0 = \) magnetic permeability of free space

A characteristic property of all waves is that the obey the principle of superposition:

"When two or more waves of the same nature travel past a point at the same time, the instantaneous amplitude there is the sum of the instantaneous amplitudes of the individual waves."

Blackbody Radiation 2.2
An ideal body that absorbs all radiation incident upon it, regardless of frequency, is called a blackbody.

The Ultra Violet Catastrophe
In blackbody spectra, the spectral distribution of energy in the radiation depends only on the temperature of the body.

Why does the blackbody spectrum have the shape it has?

Lord Rayleigh and James Jeans considered the radiation inside a cavity of absolute temperature \( T \) whose walls are perfect reflectors to be a series of standing electromagnetic waves. This is a three dimensional generalization of standing waves in a stretched string.

The condition for standing waves in such a cavity is that the path length from wall to wall, whatever the location, must be a while number of half-wavelengths, so that a node occurs at each reflecting surface. The number of independent standing waves \( G(f)df \) in the frequency interval between \( f \) and \( df \) per unit volume in the cavity turned out to be

\[ G(f)df = \frac{8\pi f^2 df}{c^3} \quad (2.1) \]

The next step is to find the average energy per standing wave. According to the theorem of equipartition of energy, a mainstay of classical physics, the average energy per degree of freedom of an entity (such as a molecule of an ideal gas) that is a number of a system of such entities in thermal equilibrium at the temperature \( T \) is \( \frac{kT}{2} \). Here \( k \) is Boltzmann’s constant

\[ k = 1.381 \times 10^{-23} \text{JK}^{-1} \]

A degree of freedom is a mode of energy possession. Thus an ideal gas molecule has three degrees of freedom, corresponding to kinetic energy of motion in three independent directions, for an average total energy of \( \frac{3kT}{2} \)

\[ I(\lambda) = \frac{2\pi c k T}{\lambda^4} \quad (2.3) \]

Even at a glance, eqn(2.3) shows that it cannot possibly be correct. As the wavelength decreases, the formula predicts that the intensity should increase as \( f^4 \). In the limit of near zero wavelengths, the intensity should go to infinity. In reality, intensity and radiation fall to zero as \( \lambda \rightarrow 0 \). This discrepancy became known as the ultra violet catastrophe of classical physics.

Planck Radiation Formula
In 1900, the German physicist Max Planck developed a formula for the total radiated intensity

\[ I(\lambda) = \frac{2\pi \hbar c^2}{\lambda^5(e^{\frac{\hbar \lambda k T}{\lambda}} - 1)} \quad (2.4) \]

At high frequencies \( e^{\frac{\hbar \lambda k T}{\lambda}} \rightarrow \infty \) which means that \( I(\lambda) \rightarrow 0 \).

At low frequencies, using the approximation \( e^x \approx 1 + x \), it agrees with empirical results also.

So, Planck’s formula was exactly right.

Next Planck had the problem of justifying eqn(2.4) in terms of physical principles. After several weeks, Planck found the answer. The oscillators in the cavity walls could not have a continuous distribution of possible energies \( \epsilon \) but must have only the specific energies
\[ \epsilon_n = nhf \quad n = 0, 1, 2... \] (2.5)

An oscillator emits radiation of frequency, \( f \), when it drops from one energy state to the next lower one, and it jumps to the next higher one when it absorbs radiation of frequency \( f \). Each discrete bundle of energy called \( hf \) is called a quantum.

With oscillator energies limited to \( nhf \), the average energy per oscillator in the cavity walls - and so per standing wave - turned out to be \( \bar{\epsilon} = kT \) as for a continuous distribution of energies, but instead

\[ \bar{\epsilon} = \frac{hf}{e^{hf/kT} - 1} \]

Photo-Electric Effect 2.3
During his experiments on electromagnetic waves, Hertz noticed that sparks occurred more readily in the air gap of his transmitter when ultraviolet light was directed at one of the metal balls. He did not follow up this observation, but others did. They soon discovered that the cause was electrons emitted when the frequency of light was sufficiently high. This phenomenon is now as the photoelectric effect and the emitted electrons are called photoelectrons. It is one of the ironies of history that the same work to demonstrate that light consists of electromagnetic waves also gave the first hint that this was not the whole story.

The photoelectric effect was studied using an evacuated tube containing two electrodes connected to a source of variable voltage, with a metal plate whose surface is irradiated as the anode. Some of the photoelectrons that emerge from this surface have enough energy to reach the cathode despite its negative polarity, and they constitute the measured current. The slower photoelectrons are repelled before they get to the cathode. When the voltage is increased to a certain value, \( V_0 \) of the order of several volts, no more photoelectrons arrive, as indicated by the current dropping to zero. This extinction voltage corresponds to the maximum photoelectron kinetic energy.

The existence of the photoelectric effect is not surprising. After all, light waves carry energy, and some of the energy absorbed by the metal may somehow concentrate on individual electrons and reappear as their kinetic energy. The situation should be like water waves dislodging pebbles from a beach. But three experimental findings show that no such simple explanation is possible:

1. Within the limits of experiment accuracy (about \( 10^{-9}s \)), there is no time interval between the arrival of light at a metal surface and the emission of photoelectrons. However, because the energy in an electromagnetic wave is supposed to be spread across the wavefronts, a period of time should elapse before an individual electron accumulated enough energy (several eV) to leave the electron. A detectable photoelectron current results when \( 10^6W/m^2 \) of electromagnetic energy is absorbed by a sodium surface. A layer of sodium 1 atom thick and 1 m\(^2\) in area contains about \( 10^{19} \) atoms, so if the incident light is absorbed in the uppermost atomic layer, each atom receives energy at an average rate of \( 10^{-25}W \). At this rate, over a month would be needed for an atom to accumulate energy of the magnitude that photoelectrons from a sodium surface are observed to have.

2. A bright light yields more photoelectrons than a dim one of the same frequency, but the electron energies remain the same. The electromagnetic theory of light, on the contrary, predicts that the more intense the light, the greater the energies of the electrons.

3. The higher the frequency of the light, the more energy the photoelectrons have. Blue light results in faster electrons than red light. At frequencies below a certain critical frequency, \( f_0 \), which is characteristic of each particular metal, no electrons are emitted. Above \( f_0 \) the photoelectrons range in energy from 0 to a maximum value (that increases linearly with increasing frequency). This observation too, cannot be explained by the electromagnetic theory of light.

Quantum Theory
In 1905, Einstein realized that the photoelectric effect could be understood if the energy in light is not spread out over wavefronts but is concentrated in small packets, or photons. Each photon of light of frequency \( f \) has the energy \( hf \), the same as Planck’s quantum energy. Planck had thought that, although energy from an electric oscillator apparently had to be given in electromagnetic waves in separate quanta of \( hf \) each, the waves them-
selves behaved exactly as in conventional wave theory. Einstein’s break with classical physics was more drastic: Energy was not only given to electromagnetic waves in separate quanta but was also carried by the waves in separate quanta.

The three experiment observations listed above follow directly from Einstein’s hypothesis. (1) Because electromagnetic waves energy is concentrated in photons and not spread out there should be no delay in the emission of photoelectrons. (2) All photons of frequency \( f \) will have the same energy, so changing the intensity of a monochromatic light beam will change the number of photoelectrons but not their energies. (3) The higher the frequency \( f \), the greater the photon energy \( hf \) and so the more energy the photoelectrons have.

What is the meaning of the critical frequency \( f_0 \) below which photoelectrons are emitted? There must be a minimum energy \( \phi \) for an electron to escape from a particular metal surface or else electrons would pop out all the time. This energy is called the work function of the metal, and is related to \( f_0 \) by the formula:

\[
\phi = hf_0 \quad (2.7)
\]

The greater the work function of a metal, the more energy is needed for an electron to leave its surface, and the higher the critical frequency for photoelectric emission to occur. According to Einstein, the photoelectric effect in a given metal should obey the equation

\[
hf = KE_{\text{max}} + \phi \quad (2.8)
\]

X-Rays 2.5

In 1895 Wilhelm Roentgen found that a highly penetrating radiation of unknown nature is produced when fast electrons impinge on matter. These x-rays were soon found to travel in straight lines, to be unaffected by electric and magnetic fields, to pass readily through opaque materials, to cause phosphorescent substances to glow, and to expose photographic plates. The faster the original electrons, the more penetrating the resulting x-rays, and the greater the number of electrons, the greater the intensity of the x-ray beam.

Not long after this discovery it became clear that x-rays are electromagnetic waves. Electromagnetic wave theory predicts that an accelerated electric charge will radiate electromagnetic waves, and a rapidly moving electron suddenly brought to rest is certainly accelerated. Radiation produced under these circumstance is given the German name bremsstrahlung ("braking radiation"). A cathode, heated by a filament through which an electric current is passed, supplies electrons by thermionic emission. The high potential difference \( V \) maintained between the cathode and the metallic target accelerates the electrons toward the latter. The fact of the target is at an angle relative to the electron beam, and the x-rays that leave the target pass through the side of the tube. The tube is evacuated to permit the electrons.

As mentioned earlier, classical electromagnetic theory predicts bremsstrahlung when electrons are accelerated, which accounts in general for the x-rays produced by an x-ray tube. However, the agreement between theory and experiment is not satisfactory in certain important respects. The x-ray spectra that result when tungsten and molybdenum targets are bombarded by electrons at several different accelerating potentials. The curves exhibit two features electromagnetic theory cannot explain:

1. In the case of molybdenum, intensity peaks occur that indicate the enhanced production of x-rays at certain wavelengths. These peaks occur at specific wavelengths for each target material and originate in rearrangements of the electron structures of the target atoms after having been disturbed by the bombarding electrons. This phenomenon will be discussed later; the important thing to note at this point is the presence of x-rays of specific wavelengths, a decidedly nonclassical effect, in addition to a continuous x-ray spectrum.

2. The x-rays produced at a given accelerating potential \( V \) vary in wavelength, but none have a wavelength shorter than a certain value \( \lambda_{\text{min}} \). Increasing \( V \) decreases \( \lambda_{\text{min}} \). At a particular \( V \), \( \lambda_{\text{min}} \) is the same for both the tungsten and molybdenum targets. Duane and Hunt found experimentally that \( \lambda_{\text{min}} \) is inversely proportional to \( V \); their precise relationship is

\[
\lambda_{\text{min}} = \frac{1.24 \times 10^{-6}}{V} \text{ m} \quad 2.12
\]

The second observation fits in with the quantum theory of radiation. Most of the electrons that
strike the target undergo number glancing collisions, with their energy simply going into heat. (This is why the targets in x-ray tubes are made from high-melting point metals such as tungsten, and a means of cooling the target is usually employed); A few electrons, though, lose most or all of their energy in single collisions with target atoms. This is the energy that becomes x-rays.

**X-Ray Diffraction 2.6**
A crystal consists of a regular array of atoms, each of which can scatter electromagnetic waves. The mechanism of scattering is straightforward. An atom in a constant electric field becomes polarized since its negatively charged electrons and positively charged nucleus experience forces in opposite directions. These forces are small compared with the forces holding the atom together, and so the result is a distorted charge distribution equivalent to an electric dipole. In the presence of the alternating electric field of an electromagnetic wave of frequency $f$, the polarization changes back and forth with the same frequency $f$. An oscillating electric dipole is thus created at the expense of some of the energy of the incoming wave. The oscillating dipole in turn radiates electromagnetic waves of frequency $f$, and these secondary waves go out in all directions except along the dipole axis. A monochromatic beam of x-rays that falls upon a crystal will be scattered in all directions inside it. However, owing to the regular arrangement of the atoms, in certain directions the scattered waves will constructively interfere with one another while in others they will destructively interfere. The atoms in a crystal may be thought of as defining families of parallel plates, with each family having a characteristic separation between its component planes. This analysis was suggested in 1913 by W.L. Bragg, in honour of whom the above planes are called Bragg planes.

**Compton Effect 2.7**
According to the quantum theory of light, photons behave like particles except for their lack of rest mass. How far can this analogy be carried? For instance, can we consider a collision between a photon and an electron as if both were billiard balls? Imagine such a collision. An x-ray photon strikes an electron (assumed to be initially at rest in the laboratory coordinate system) and is scattered away from its original direction of motion by an angle $\phi$ while the electron receives an impulse and begins to move away from the photon’s original direction of motion at an angle $\theta$ where $\phi$ and $\theta$ are most likely not equal. We can think of the photon as losing an amount of energy in the collision that is the same as the kinetic energy (KE) gained by the electrons, although actually separate photons are involved. If the initial photon has the frequency $f$ associated with it, the scatter photon has the lower frequency $f'$ where

$$\text{loss in photon energy} = \text{gain in electron energy}$$

$$hf - hf' = KE$$  \hspace{1cm}  (2.14)

Recall

$$E = pc$$

Since the energy of a photon is $hf$, it momentum is

$$p = \frac{E}{c} = \frac{hf}{c}$$  \hspace{1cm}  (2.15)

Momentum, unlike energy, is a vector quantity that incorporates direction as well as magnitude, and in the collision momentum must be conserved in each of the two mutually perpendicular directions. The directions chosen here are that of the original photon and one perpendicular to it in the plane containing the electron and the scattered photon. The initial photon momentum is $\frac{hf}{c}$, the scattered photon momentum is $\frac{hf'}{c}$ and the initial and final electron momenta are 0 and $p$ respectively. In the original photon direction:

Initial momentum = final momentum

$$\frac{hf}{c} + 0 = \frac{hf'}{c} \cos \phi + p \cos \theta$$  \hspace{1cm}  (2.16)

and perpendicular to this direction

Initial momentum = final momentum

$$0 = \frac{hf'}{c} \sin \phi - p \sin \theta$$  \hspace{1cm}  (2.17)

The angle $\phi$ is that between the directions of the initial and scattered photons, and $\theta$ is that between the directions of the initial photon and the recoil photon. From eqns (2.14), (2.16) and (2.17) we can find a formula that relates the wavelength difference between initial and scattered photons with
the angle $\phi$ between their directions, both of which are readily measurable quantities.

The first step is to multiply eqns (2.16) and (2.17) by $c$ and rewrite them as

\[
\begin{align*}
pc \cos \theta &= hf - hf' \cos \phi \\
pc \sin \theta &= hf' \sin \phi
\end{align*}
\]

By squaring each of these equations and adding the new ones together, the angle $\theta$ is eliminated, leaving

\[
p^2c^2 = (hf)^2 - 2(hf)(hf') \cos \phi + (hf')^2 \tag{2.18}
\]

Next we equate the two expressions for the total energy of a particle

\[
\begin{align*}
E &= KE + m_0c^2 \\
E &= \sqrt{m_0c^4 + p^2c^2} \\
(KE + m_0c^2)^2 &= m_0c^4 + p^2c^2 \\
p^2c^2 &= KE^2 + 2m_0c^2KE
\end{align*}
\]

Since

\[
KE = hf - hf'
\]

we have

\[
p^2c^2 = (hf)^2 - 2(hf)(hf') + (hf')^2 + 2m_0c^2(hf - hf') \tag{2.19}
\]

Substituting this value of $p^2c^2$ in eqn (2.18), we finally obtain

\[
2m_0c^2(hf - hf') = 2(hf)(hf')(1 - \cos \phi) \tag{2.20}
\]

This relationship is simple when expressed in terms of wavelength $\lambda$. Dividing eqn (2.20) by $2h^2c^2$

\[
\frac{m_0c}{h} \left( \frac{v}{c} - \frac{v'}{c} \right) = \frac{v v'}{c c} (1 - \cos \phi)
\]

and so, since $\frac{v}{c} = \frac{1}{\lambda}$ and $\frac{v'}{c} = \frac{1}{\lambda'}$

\[
\frac{m_0c}{h} \left( \frac{1}{\lambda} - \frac{1}{\lambda'} \right) = \frac{1 - \cos \phi}{\lambda \lambda'}
\]

\[
\lambda' - \lambda = \frac{h}{m_0c} (1 - \cos \phi) \tag{2.21}
\]

If you detect any errors in these notes pls contact me at pwalsh@mcmaths.tcd.ie and I will try to amend them.

Notes after this point currently being drafted.