

Analysis Problems #5

1. Show that the polynomial $f(x) = x^3 - 4x^2 - 3x + 1$ has exactly one root in $[0, 2]$.
2. Suppose $\log x$ is a function such that $(\log x)' = \frac{1}{x}$ and $\log 1 = 0$. Show that

$$\log x \leq x - 1 \quad \text{for all } x > 0.$$

3. Suppose that $x > y > 0$. Use the mean value theorem to show that

$$1 - \frac{y}{x} < \log x - \log y < \frac{x}{y} - 1.$$

4. Suppose that f is twice differentiable with $f(0) = f'(0) = 0$ and

$$f''(x) + f(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

Show that $f(x)^2 + f'(x)^2 = 0$ for all $x \in \mathbb{R}$. Deduce that $f(x) = 0$ for all $x \in \mathbb{R}$.

5. Let $n \in \mathbb{N}$ be fixed. Show that $x^n \log x \geq -\frac{1}{ne}$ for all $x > 0$.