Linear algebra II
Tutorial problems #7

1. Consider the bilinear form on $\mathbb{R}^2$ defined by the formula

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 4x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2.$$

Find a basis B of $\mathbb{R}^2$ such that the matrix of the form with respect to B is diagonal. Hint: find the matrix with respect to the standard basis and its eigenvectors.

2. An $n \times n$ matrix A is called orthogonal, if $A^tA = I$. Show that left multiplication by an orthogonal matrix preserves the dot product of two vectors, namely

$$\langle A\boldsymbol{x}, A\boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Conclude that $A\boldsymbol{x}$ and $\boldsymbol{x}$ have the same length, if A is orthogonal.

3. We will soon show that every $n \times n$ real symmetric matrix is diagonalizable with real eigenvalues. Prove this statement when $n = 2$. Hint: show that

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

has two distinct, real eigenvalues for all $a, b, c \in \mathbb{R}$.