Find the derivative of the function $y = \arctan x$.

Solution: Since $y = \arctan x = \tan^{-1} x$, we have

$$\begin{aligned} \tan y &= \tan(\tan^{-1} x), \\ \tan y &= x. \end{aligned}$$

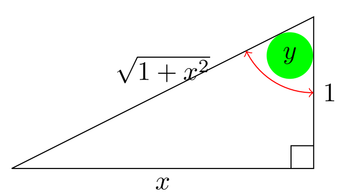
The derivative of $\tan y$ is:

$$\begin{aligned} \frac{d}{dy} \tan y &= \frac{d}{dy} \frac{\sin y}{\cos y} \\ &= \frac{\cos^2 y + \sin^2 y}{\cos^2 y} \\ &= \frac{1}{\cos^2 y} = \sec^2 y. \end{aligned}$$

Take the derivative of both sides of the equation $\tan y = x$ to get

$$\begin{aligned} \frac{d}{dx} \tan y &= \frac{d}{dx} x, \\ \text{(Chain Rule)}\ \frac{d}{dy}(\tan y)\ \frac{dy}{dx} &= 1, \\ \left(\frac{1}{\cos^2 y}\right) \frac{dy}{dx} &= 1, \\ \frac{dy}{dx} &= \cos^2 y = \cos^2(\arctan x) \end{aligned}$$

So the derivative of the function $y = \arctan x$ is $\frac{dy}{dx} = \cos^2(\arctan x)$. This formula gives a correct answer but it can be simplified. Consider the right triangle:



$\tan y = x,\ y = \arctan x$

$\cos y = \frac{1}{\sqrt{1+x^2}}$

Then $\cos^2 y = (\frac{1}{\sqrt{1+x^2}})^2 = \frac{1}{1+x^2}$. Thus if $y = \arctan x$ then

$$\begin{aligned} \frac{dy}{dx} &= \cos^2 y \\ &= \frac{1}{1 + x^2}. \end{aligned}$$