Triangular Matrices & the Reduced Row Echelon Form

1. Prove that the product of two upper triangular matrices is an upper triangular matrix. (An n-by-n matrix $A = [a_{ij}]_{i,j=1}^n$ is *upper triangular* if all elements below the main diagonal are 0, i.e. if $i > j$ then $a_{ij} = 0$.)
Proof: Let $A = [a_{ij}]_{i,j=1}^n$ and $B = [b_{ij}]_{i,j=1}^n$ be n-by-n upper triangular matrices. Let $C = AB = [c_{ij}]_{i,j=1}^n$ where $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$. To show that C is upper triangular, we must show that: if $i > j$ then $c_{ij} = 0$.

Fix $i, j \in \{1, 2, \ldots, n\}$ with $i > j$. Then $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$ and:

$$\text{if } i > k \text{ then } a_{ik} = 0 \;, \qquad \text{if } k > j \text{ then } b_{kj} = 0.$$

Hence $a_{ik} b_{kj} = 0$ for all $k \in \{1, 2, \ldots, n\}$. So $c_{ij} = 0$.

2. There are three elementary row operations one can perform on matrices:

1. Multiplying a row by a non-zero constant.
2. Switching two rows.
3. Adding a row to another row.

These three elementary row operations can be used to convert a matrix into the reduced row echelon form. A matrix is in the reduced row echelon form if it satisfies the conditions:

- The first non-zero entry in a non-zero row is one (call it the 'leading one').
- The leading one in a row is to the right of the leading one in the row above.
- Rows of zeros are at the bottom of the matrix.
- Each column that contains a leading one has all other entries zero.

Example:

$$\begin{bmatrix} \mathbf{1} & * & 0 & 0 & * & * & 0 & * & 0 \\ 0 & 0 & \mathbf{1} & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & \mathbf{1} & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ represents numbers.

To show that the reduced row echelon form of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad - bc \neq 0$, is the identity matrix I_2, consider two cases:
Case 1: $a = 0$. Then $b \neq 0$ and $c \neq 0$. We have:
$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & b \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$

Case 2: $a \neq 0$. Then we have:
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$

The Definitions of Limits and Continuity

$$\boxed{\lim_{x\to a} f(x) = L} \Leftrightarrow \boxed{\forall\, \epsilon > 0 \;\; \exists\, \delta > 0 \;\; \forall\, x \in \mathbb{R} \quad \Big[(0 < |x-a| < \delta) \Rightarrow |f(x) - L| < \epsilon\Big]}$$

$$\boxed{\lim_{x\to a^-} f(x) = L} \Leftrightarrow \boxed{\forall\, \epsilon > 0 \;\; \exists\, \delta > 0 \;\; \forall\, x \in \mathbb{R} \quad \Big[(0 < a-x < \delta) \Rightarrow |f(x) - L| < \epsilon\Big]}$$

$$\boxed{\lim_{x\to a^+} f(x) = L} \Leftrightarrow \boxed{\forall\, \epsilon > 0 \;\; \exists\, \delta > 0 \;\; \forall\, x \in \mathbb{R} \quad \Big[(0 < x-a < \delta) \Rightarrow |f(x) - L| < \epsilon\Big]}$$

Let f be a function and let a belong to the domain of f (so that $f(a)$ makes sense).

- f is *continuous* at a if

  $$\forall\, \epsilon > 0 \;\; \exists\, \delta > 0 \;\; \forall\, x_{\text{ in the domain}} \quad \Big[|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon\Big].$$

  (It follows that f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$.)

- f is *left-continuous* at a if $\lim_{x\to a^-} f(x) = f(a)$.

- f is *right-continuous* at a if $\lim_{x\to a^+} f(x) = f(a)$.