Mathematics 1123
Sample Test
January '12

Answer <u>all</u> questions

1 (i) Define what is meant by a function from a set X to a set Y.

A function from a set X to a set Y is a relation f between the elements of X and Y (if x is f-related to y, write $f(x) = y$) such that:

1. $\forall x \in X \quad \exists y \in Y \qquad f(x) = y$;
2. $\forall x \in X \quad \forall y, z \in Y \qquad$ If $f(x) = y$ and $f(x) = z$, then $y = z$.

(ii) Define what it means for $f\colon X \to Y$ to be onto (surjective).

A function $f\colon X \to Y$ is onto (or surjective) if
$\forall y \in Y \quad \exists x \in X \quad$ with $\ f(x) = y$.

(iii) What does the vertical line test of a graph tell?

The vertical line test determines whether the conditions 1 and 2 (see **1 (i)**) hold.

(iv) Define $\lim\limits_{x\to a^-} f(x) = L$, $\lim\limits_{x\to\infty} f(x) = L$.

$\lim\limits_{x\to a^-} f(x) = L \Leftrightarrow \forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in \mathbb{R} \quad \Big(0 < a - x < \delta\Big) \Rightarrow |f(x) - L| < \epsilon$,

$\lim\limits_{x\to\infty} f(x) = L \Leftrightarrow \forall \epsilon > 0 \ \exists N > 0 \ \forall x \in \mathbb{R} \quad \Big(x > N\Big) \Rightarrow |f(x) - L| < \epsilon$.

(v) If $\lim\limits_{x\to a^-} f(x) = L_1$ and $\lim\limits_{x\to a^+} f(x) = L_2$, prove $L_1 \neq L_2 \Rightarrow \lim\limits_{x\to a} f(x)$ does not exist.

The ordinary limit $\lim\limits_{x\to a} f(x)$ exists and is equal to $L \Leftrightarrow$ the left- and right-hand limits $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist and both are equal to L. Thus: if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ exist and are equal to L_1 and L_2, respectively, but $L_1 \neq L_2$, then the ordinary limit $\lim\limits_{x\to a} f(x)$ does not exist.

2 (i) Let $f(x) = -x$ if $x \le 0$, $f(x) = x^2$ if $x \ge 0$. Find $f'(0)$ or show it does not exist.

The limit $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{f(x)}{x} = \lim\limits_{x\to 0} \begin{cases} -1 & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$ does not exist (because the left-hand limit is -1 and the right-hand limit is 0), thus $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist.

(ii) Let $f(x) = x \sin \frac{1}{x}$ if $x \neq 0$, $f(0) = 0$. Find $f'(0)$ or show it does not exist. Is $f(x)$ continuous at $x = 0$?

The limit $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{f(x)}{x} = \lim\limits_{x\to 0} \dfrac{x \sin \frac{1}{x}}{x} = \lim\limits_{x\to 0} \sin \dfrac{1}{x}$ does not exist (because the function $\sin \frac{1}{x}$ $(x \neq 0)$ keeps oscillating back and forth between -1 and 1), thus $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist. Since

$-1 \leq \sin\frac{1}{x} \leq 1$, $-x \leq x\sin\frac{1}{x} \leq x$, we have $\lim_{x\to 0} x\sin\frac{1}{x} = 0 = f(0)$, so $f(x)$ is continuous at $x = 0$.

(iii) If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, prove $\lim_{x\to a} f(x)g(x) = L_1L_2$.

Assume $f, g\colon \mathbb{R} \to \mathbb{R}$ are functions; $a, L_1, L_2 \in \mathbb{R}$; and $\lim_{x\to a} f(x) = L_1$, $\lim_{x\to a} g(x) = L_2$. Fix $\epsilon > 0$. Since $\lim_{x\to a} f(x) = L_1$, we have

$\exists\delta_0 > 0\ \forall x \in \mathbb{R}$ with $0 < |x-a| < \delta_0 : \quad |f(x)| - |L_1| \leq |f(x) - L_1| < \epsilon$, thus

$$\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta_0 : \qquad |f(x)| < |L_1| + \epsilon.$$

Again, since $\lim_{x\to a} f(x) = L_1$, we can choose (possibly) another $\delta_1 > 0$ such that

$$\forall x \in \mathbb{R} \text{ with } 0 < |x-a| < \delta_1 : \qquad |f(x) - L_1| < \frac{\epsilon}{2\,|L_2|}.$$

Since $\lim_{x\to a} g(x) = L_2$, we have
$\exists\delta_2 > 0\ \ \forall x \in \mathbb{R}$ with $0 < |x-a| < \delta_2 : \quad |g(x) - L_2| < \frac{\epsilon}{2(|L_1|+\epsilon)}$.
Take $\delta = \text{minimum}\,\{\delta_0, \delta_1, \delta_2\}$. Then $\forall x \in \mathbb{R}$ with $0 < |x-a| < \delta$ we have

$$\begin{aligned}
|f(x)g(x) - L_1L_2| &= |f(x)g(x) - f(x)L_2 + f(x)L_2 - L_1L_2| \\
&= \Big|f(x)\Big(g(x) - L_2\Big) + \Big(f(x) - L_1\Big)L_2\Big| \\
&\leq \Big|f(x)\Big(g(x) - L_2\Big)\Big| + \Big|\Big(f(x) - L_1\Big)L_2\Big| \\
&= |f(x)|\,|g(x) - L_2| + |f(x) - L_1|\,|L_2| \\
&< (|L_1| + \epsilon)\frac{\epsilon}{2(|L_1| + \epsilon)} + \frac{\epsilon}{2\,|L_2|}\,|L_2| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.
\end{aligned}$$

This proves $\lim_{x\to a} f(x)g(x) = L_1L_2$.

(iv) Find the linear approximation and the quadratic approximation to $\sin(.05)$.

Linear approximation: $\sin(.05) \approx .05$
Quadratic approximation: $\sin(.05) \approx .05$

3 (i) Find $\frac{dy}{dx}$ if

(a) $y = xe^x \sin x$

$$\begin{aligned}
\frac{dy}{dx} &= (xe^x)'\sin x + xe^x(\sin x)' \\
&= (e^x + xe^x)\sin x + xe^x\cos x \\
&= e^x(1+x)\sin x + xe^x\cos x \\
&= e^x\Big((1+x)\sin x + x\cos x\Big)
\end{aligned}$$

(b) $y = \ln\Big(\big(\sin(x^2+1)\big)^3\Big)$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\big(\sin(x^2+1)\big)^3}\Big(\big(\sin(x^2+1)\big)^3\Big)' \\
&= \frac{1}{\big(\sin(x^2+1)\big)^3} \cdot 3\big(\sin(x^2+1)\big)^2 \cdot \big(\sin(x^2+1)\big)' \\
&= \frac{1}{\big(\sin(x^2+1)\big)^3} \cdot 3\big(\sin(x^2+1)\big)^2 \cdot \cos(x^2+1) \cdot (x^2+1)' \\
&= \frac{1}{\big(\sin(x^2+1)\big)^3} \cdot 3\big(\sin(x^2+1)\big)^2 \cdot \cos(x^2+1) \cdot 2x \\
&= \frac{1}{\sin(x^2+1)} \cdot 3\cos(x^2+1) \cdot 2x \\
&= 6x\frac{\cos(x^2+1)}{\sin(x^2+1)} \\
&= 6x\cot(x^2+1)
\end{aligned}$$

(c) $x^2y + yx^2 = 1$

$$(x^2+x^2)y = 1, \quad y = \frac{1}{2\,x^2}, \quad \frac{dy}{dx} = \frac{-1}{x^3}$$

(d) $x = t^2+1,\ y = 2t+1$

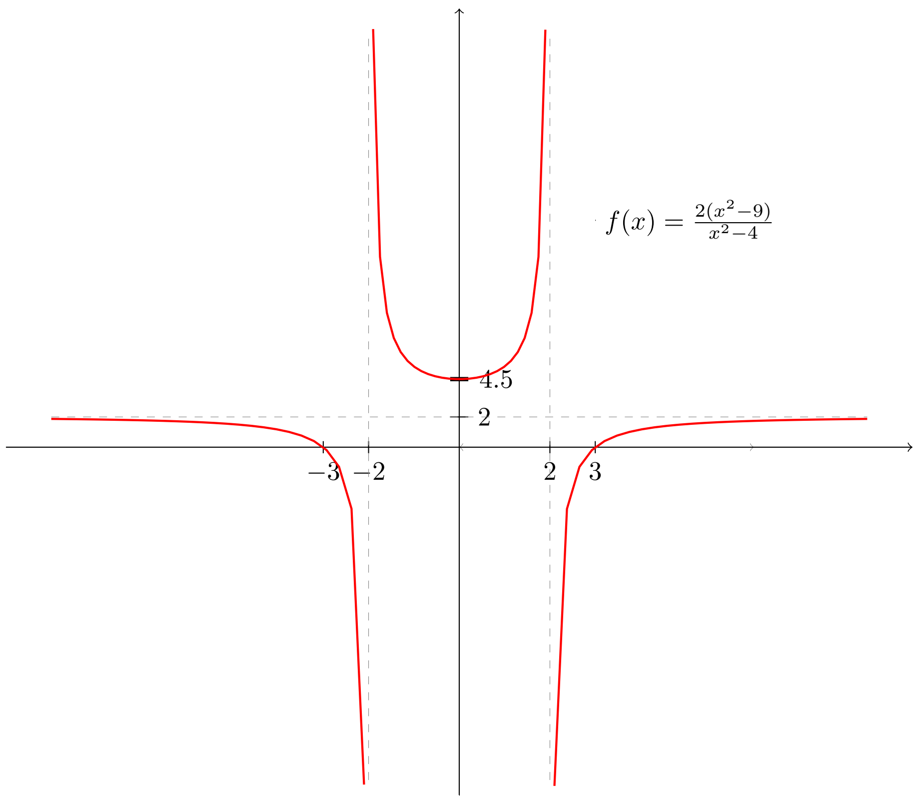
$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2, \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{t}$$

(ii) Let $f(x) = \frac{2(x^2-9)}{x^2-4}$, find where $f(x)$ increases, decreases, is concave up, concave down, local extrema, points of inflection. Hence sketch this function.

The derivatives of f are: $f'(x) = \frac{20x}{(x^2-4)^2}$, $f''(x) = \frac{-20(3x^2+4)}{(x^2-4)^3}$. The table below shows where $f(x)$ increases, decreases.

x	$(-\infty,-3)$	-3	$(-3,-2)$	-2	$(-2,0)$	0	$(0,2)$	2	$(2,3)$	3	$(3,+\infty)$
$f(x)$	positive $\searrow$	0	negative $\searrow$	asympt	positive $\searrow$	4.5	positive $\nearrow$	asympt	negative $\nearrow$	0	positive $\nearrow$
$f'(x)$	$-$		$-$		$-$	0	$+$		$+$		$+$
$f''(x)$	$-$	$-$	$-$		$+$	$\frac{5}{4} > 0$	$+$		$-$	$-$	$-$

This table further shows that f has a local minimum at $x = 0$, $f(0) = 4.5$; that f is concave up on $(-2,2)$ (since $f''(x) > 0$ for $x \in (-2,2)$); that f is concave down on $(-\infty,-2)$ and $(2,+\infty)$ (since $f''(x) < 0$ for $x \in (-\infty,-2)\cup(2,+\infty)$; and that f does not have a point of inflection.
With this, we can sketch the graph of f.



Graph of $f(x) = \frac{2(x^2-9)}{x^2-4}$

4 (i) State and prove Rolle's Theorem.

Theorem (Rolle's theorem): Let $a, b \in \mathbb{R}$, $a < b$, and let $f \colon [a, b] \to \mathbb{R}$ be a function. If

f is continuous on $[a, b]$,

f differentiable (a, b),

and $f(a) = f(b)$,

then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Proof: If f is a constant function on $[a, b]$, we have $f'(c) = 0$ for all $c \in (a, b)$, hence the conclusion of the theorem is obviously true.

Suppose f is not constant on $[a, b]$. Then there are $x_1, x_2 \in [a, b]$ with

$$f(x_1) = \inf f([a, b]), \qquad f(x_2) = \sup f([a, b]),$$

and at least one of these points, x_1, x_2, must belong to (a, b) (since f is not constant). So f has a local extremum at $c \in \{x_1, x_2\} \cap (a, b)$, and $f'(c) = 0$.

(ii) State the Mean Value Theorem and prove that if $f'(x) = 0$, all x in $[a, b]$, then $f(x) =$ constant on $[a, b]$.

Theorem (The Mean Value Theorem): Let $a, b \in \mathbb{R}$, $a < b$, and let $f \colon [a, b] \to \mathbb{R}$ be a function. If

f is continuous on $[a, b]$ and

f differentiable (a, b),

then $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

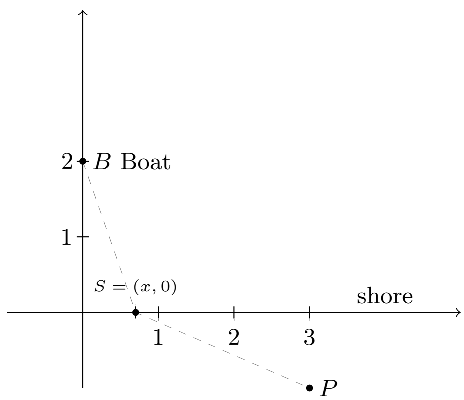
As an application of the Mean Value Theorem, we can prove that: If $f\colon [a,b] \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for all $x \in [a,b]$, then f is constant on $[a,b]$. Indeed, the Mean Value Theorem implies that for any $a_0 \in [a,b)$

$$f'(c) = \frac{f(b) - f(a_0)}{b - a_0}$$

for some $c \in (a_0, b)$. But $f'(c) = 0$. So $f(b) = f(a_0)$ for all $a_0 \in [a,b)$, which implies that f is constant on $[a,b]$.

(iii) A Boat B is 2 miles from the shore. The person in the boat wants to get to a point that is 1 mile inland from a point three miles down the shore. If the person can row at 2 miles per hour and walk at 4 miles per hour, what route should they take?

Make a diagram:



The person in the boat B is 2 miles from (the nearest point on) the shore, and wants to reach the point P in the least time. Let t_r be the rowing time, t_w the walking time, and let $S = (x, 0)$ ($x \in [0, 3]$) be the 'optional' point (on the shore) the person should row towards first. Then we have:

$BS = \sqrt{x^2 + 4}$,

$SP = \sqrt{(3 - x)^2 + 1}$,

Row BS in $t_r = \frac{BS}{2} = \frac{\sqrt{x^2+4}}{2}$

Walk SP in $t_w = \frac{SP}{4} = \frac{\sqrt{(3-x)^2+1}}{4}$

Total time is the sum of those: $t = t_r + t_w = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{(3-x)^2+1}}{4}$

Now $\frac{dt}{dx} = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{(3-x)^2+1}}$. Set that equal to zero to get:

$$\frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{(3-x)^2+1}} = 0 \quad \Leftrightarrow \quad x = 1$$

So the person in the boat B should first row toward $S = (1, 0)$ in order to reach point P in the least time.

5 (i) Use Riemann sums to derive the formula for the arc length of $y = f(x)$ from $x = a$ to $x = b$.

Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous and differentiable ($a, b \in \mathbb{R}$, $a < b$).
Pick $x_0, x_1, \ldots, x_n \in [a, b]$ with $a = x_0 < x_1 < \cdots < x_n = b$.
Consider the segment lines between
$(x_0, f(x_0))$ and $(x_1, f(x_1))$,
$(x_1, f(x_1))$ and $(x_2, f(x_2))$, ...
$(x_{n-1}, f(x_{x-1}))$ and $(x_n, f(x_n))$.
This polygonal path approximates the curve $y = f(x)$ from $x = a$ to $x = b$, and its length approximates the arc length of $y = f(x)$ from $x = a$ to $x = b$.
For all $k \in \{1, 2, \ldots, n\}$, the length of the k-th line segment is

$$\ell_k = \sqrt{\Big(x_k - x_{k-1}\Big)^2 + \Big(f(x_k) - f(x_{k-1})\Big)^2},$$

and, by the Mean Value Theorem, there exists $c_k \in [x_{k-1}, x_k]$ such that

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}, \text{ so } \qquad f(x_k) - f(x_{k-1}) = f'(c_k)\big(x_k - x_{k-1}\big).$$

Thus

$$\begin{aligned} \ell_k &= \sqrt{\Big(x_k - x_{k-1}\Big)^2 + \Big(f'(c_k)\big(x_k - x_{k-1}\big)\Big)^2} \\ &= \sqrt{1 + f'(c_k)}\big(x_k - x_{k-1}\big). \end{aligned}$$

Now, the length of the entire polygonal path is the sum of the lengths of the individual line segments,

$$\sum_{k=1}^{n} \ell_k = \sum_{k=1}^{n} \sqrt{1 + f'(c_k)}\big(x_k - x_{k-1}\big),$$

and it has the form of a Riemann sum.
Increasing n, and the number of points $x_0, x_1, \ldots, x_n \in [a, b]$, such that $\sup\{|x_k - x_{k-1}| : k \in \{1, \ldots, n\}\} \to 0$, we have that the arc length ℓ of $y = f(x)$ from $x = a$ to $x = b$ is given by:

$$\ell \;=\; \lim_{n\to\infty} \sum_{k=1}^{n} \ell_k \;=\; \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

(ii) If

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

what is $\int_a^b f(x)dx$? Explain your reasoning.

(1) This function is not Riemann integrable, it is a result due to Dirichlet.[1] To prove this, let $P = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of $[a, b]$ (meaning $a = x_0 < x_1 < x_2 < \cdots < x_n = b$). From any subinterval $[x_{k-1}, x_k]$

[1] P. G. Lejeune-Dirichlet, Werke, vol. 1, Berlin, **1889** (p. 132).

pick a rational number q_k and an irrational number r_k. Then, the lower Riemann sum is

$$L(P,f) = \sum_{k=1}^{n}(x_k - x_{k-1})f(r_k) = 0,$$

and the upper Riemann sum is

$$U(P,f) = \sum_{k=1}^{n}(x_k - x_{k-1})f(q_k) = x_n - x_0 = b - a > 0.$$

Hence

$$\overline{\int_a^b f(x)dx} \;=\; \inf\{U(P,f)\} \;=\; b-a \;>\; 0$$

and

$$\underline{\int_a^b f(x)dx} \;=\; \sup\{L(P,f)\} \;=\; 0.$$

It follows, therefore, that f is not Riemann integrable.

(iii) Integrate

(a) $\int x \ln x^2 dx$

$$\begin{aligned}\int x\ln(x^2)dx &= \frac{1}{2}\int \ln(u)du \quad \text{(letting } x^2 = u,\quad 2xdx = du,\quad xdx = \tfrac{1}{2}du\text{)}\\ &= \frac{1}{2}x^2\Big(\ln(x^2) - 1\Big) + C \quad \text{(where } C \text{ is the integration constant)}\end{aligned}$$

(b) $\int x^2 \ln x dx$

$$\begin{aligned}\int x^2\ln(x)dx &= uv - \int uv' \quad \Big(\text{letting } u' = x^2,\ v = \ln(x),\quad \text{we have}\quad u = \tfrac{1}{3}x^3,\ v' = \tfrac{1}{x}\Big)\\ &= \frac{1}{3}x^3\ln(x) - \frac{1}{3}\int x^2 dx\\ &= \frac{1}{3}x^3\Big(\ln(x) - \frac{1}{3}\Big) + C\end{aligned}$$

(c) $\int \frac{x^2+1}{x+2}dx$

$$\begin{aligned}\int \frac{x^2+1}{x+2}dx &= \int \Big(x + \frac{5}{x+2} - 2\Big)dx\\ &= \int xdx + \int \frac{5}{x+2}dx - 2\int dx\\ &= \frac{x^2}{2} + 5\ln(x+2) - 2x + C\end{aligned}$$

(d) $\int \frac{x+2}{x^2+1}dx$

$$\begin{aligned}\int \frac{x+2}{x^2+1}dx &= \int \Big(\frac{x}{x^2+1} + \frac{2}{x^2+1}\Big)dx \\ &= \int \frac{x}{x^2+1}dx + 2\int \frac{1}{x^2+1}dx \\ &= \frac{1}{2}\ln(x^2+1) + C_1 \ + \ 2\int \frac{1}{x^2+1}dx \\ &= \frac{1}{2}\ln(x^2+1) + C_1 \ + \ 2\int \frac{1}{\tan^2(u)+1}\cdot\frac{1}{\cos^2(u)}du \ \ (x = \tan(u)) \\ &= \frac{1}{2}\ln(x^2+1) + C_1 \ + \ 2\tan^{-1}(x) + C_2\end{aligned}$$

(e) $\int \frac{x+2}{(x-1)^2(x-2)}dx$

$$\begin{aligned}\int \frac{x+2}{(x-1)^2(x-2)}dx &= \int \Big(\frac{4}{x-2} - \frac{4}{x-1} - \frac{3}{(x-1)^2}\Big)dx \\ &= \int \frac{4}{x-2}dx - \int \frac{4}{x-1}dx - \int \frac{3}{(x-1)^2}dx \\ &= 4\ln(x-2) - 4\ln(x-1) + \frac{3}{x-1} + C\end{aligned}$$

(f) $\int \cos^2 x \sin^2 x dx$

$$\begin{aligned}\int \cos^2(x)\sin^2(x)dx &= \int \Big(\cos(x)\sin(x)\Big)^2 dx \\ &= \int \Big(\frac{1}{2}\sin(2x)\Big)^2 dx \ \ \Big(\text{since } \sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)\Big) \\ &= \frac{1}{4}\int \sin^2(2x)dx \\ &= \frac{1}{4}\int \frac{1-\cos(4x)}{2}dx \ \ \Big(\text{since } \sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2}\Big) \\ &= \frac{1}{8}\int dx - \frac{1}{8}\int \cos(4x)dx \\ &= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C\end{aligned}$$

6 (i) Prove that the volume of a pyramid with square base is 1/3 h B, where h is the height and B is the area of the base.

Consider $f\colon [0,h] \to \mathbb{R}$ defined by $f(x) = \frac{(h-x)^2}{h^2}B$. Then, for any $x \in [0,h]$, $f(x)$ is the area of a cross-section parallel to the base at distance x from the base. Hence the volume V is

$$\begin{aligned} V &= \int_0^h f(x)dx \\ &= \int_0^h \frac{(h-x)^2}{h^2} B dx \\ &= \frac{B}{h^2}\int_0^h (h-x)^2 dx \\ &= \frac{1}{3}hB. \end{aligned}$$

(ii) Find the volume of the solid formed by revolving the region bounded by $y = x^2+1$, $y=0$, $x=0$ and $x=1$, about the y-axis. Do it first by disks and then by cylindrical shells.

Disks: $V = \int_0^1 \pi dy + \int_1^2 \pi(1-(\sqrt{y-1})^2)dy = \frac{1}{3}\pi$.
Cylindrical shells: $V = \int_0^1 2\pi x(x^2+1) = \frac{1}{3}\pi$.

(iii) Define hyperbolic sine and cosine. Why are they called hyperbolic?

Hyperbolic sine: For any $x \in \mathbb{R}$, the hyperbolic sine of x, denoted $\sinh(x)$, is defined by $\sinh(x) = \frac{e^x - e^{-x}}{2}$.
Hyperbolic cosine: For any $x \in \mathbb{R}$, the hyperbolic cosine of x, denoted $\cosh(x)$, is defined by $\cosh = \frac{e^x+e^{-x}}{2}$
Why are they called hyperbolic? Reason: The equation of the unit circle is $u^2+v^2=1$. For any real number x, we have $\cos^2(x)+\sin^2(x)=1$; thus the point $(\cos(x), \sin(x))$ lies on the circle $u^2+v^2=1$. Now consider the hyperbolic sine and cosine. For any real number x, we have $\cosh^2(x) - \sinh^2(x) = 1$; thus the point $(\cosh(x) - \sinh(x))$ lies on the curve $u^2 - v^2 = 1$, which is a hyperbola. This explains the name hyperbolic sine and cosine.