L'Hôpital's Rule states that, under certain conditions, the limit of $\frac{f(x)}{g(x)}$ is the limit of $\frac{f'(x)}{g'(x)}$. Specifically:

Theorem (L'Hôpital's Rule): Let $-\infty \leq a < b \leq +\infty$, let $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ be functions, and let $x_0 \in \{a,b\}$. Suppose that

(1) f and g are differentiable on (a,b) with $g'(x) \neq 0$ for all x in (a,b), and
(2) $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{-\infty, +\infty\}$.

If

(3) $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = 0$, or
(4) $\lim_{x\to x_0} g(x) \in \{-\infty, +\infty\}$,

then there exists $c \in (a,b)$ such that $g(x) \neq 0$ for all x in $(a,b) \setminus \{c\}$ and

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L.$$

Application Examples:

(i) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\sin x} = \lim_{x\to 0} \frac{e^x}{\cos x} = 1.$

(ii) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}.$

(iii) $\displaystyle\lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{-\frac{1}{x^2}}{-e^{\frac{1}{x}}\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{1}{e^{\frac{1}{x}}} = 0.$

(iv) $\displaystyle\lim_{x\to +\infty} \frac{x}{e^x} = \lim_{x\to +\infty} \frac{1}{e^x} = 0.$

(v) $\displaystyle\lim_{x\to 0^+} x^x$ is an indeterminate form of type 0^0. By taking logarithms we can reduce the problem to computing $\displaystyle\lim_{x\to 0^+} x\log x$ which is 0. Thus

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\log x} = e^0 = 1.$$

(vi) $\displaystyle\lim_{x\to +\infty} (1+\frac{1}{x})^x$ is of the form 1^∞. Taking logarithms, we reduce to evaluating $\displaystyle\lim_{x\to +\infty} x\log(1+\frac{1}{x})$. This is of type $\infty \cdot 0$, so we can convert it to

$$\lim_{x\to +\infty} \frac{\log(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to +\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x\to +\infty} \frac{1}{1+\frac{1}{x}} = 1$$

so

$$\lim_{x\to +\infty} (1+\frac{1}{x})^x = \lim_{x\to +\infty} e^{\log(1+\frac{1}{x})^x} = \lim_{x\to +\infty} e^{x\log(1+\frac{1}{x})} = e^1 = e.$$

Failure of L'Hôpital's Rule: In applying L'Hôpital's Rule, it must be checked that the hypotheses apply. For example, one might argue

$$\lim_{x\to 0} \frac{x^2}{x^2+\sin x} = \lim_{x\to 0} \frac{2x}{2x+\cos x} = \lim_{x\to 0} \frac{2}{2-\sin x} = \frac{2}{2} = 1.$$

However, the second limit is not an indeterminate form and the second application of L'Hôpital's Rule is not justified. The second limit can be evaluated directly to be 0, and thus the first limit is also 0.