Maths 1124 – Assignment 5

(Q1.) Write out all the details to prove if $a = l.u.b\left\{x \,:\, x^2 < 2\right\}$, then $a^2 = 2$.
Solution: Let $S = \left\{x \in \mathbb{Q} \,:\, x^2 < 2\right\}$, and let $a = l.u.b\ S$. Notice that $a^2 < 2$ and $a^2 > 2$ both lead to contradictions (so we must have $a^2 = 2$).

(Q2.) The integers $\mathbb{Z}$ are not well ordered[1] for the usual order. Give them a different order in which they are well ordered.
Solution: The set of all integers is not well ordered if arranged in the usual order

$$\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots \ .$$

However, if arranged in the order

$$0, 1, -1, 2, -2, 3, -3, \ldots,$$

it is well ordered (and 0 is a first element of $\mathbb{Z}$ when arranged in this order).
In fact, if $f \colon \mathbb{Z} \to \mathbb{N}$ is a bijection, define

$$\forall a, b \in \mathbb{Z} : \qquad a \leq_f b \ \Leftrightarrow\ f(a) \leq f(b).$$

Then $\mathbb{Z}$ with $\leq_f$ is well ordered.

(Q3.) If $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} a_n = b$, show $a = b$.
Solution: We have

$$\begin{aligned} 0 &= \lim_{n\to\infty}(a_n - a_n) \\ &= \lim_{n\to\infty}(a_n) - \lim_{n\to\infty}(a_n) \\ &= a - b, \end{aligned}$$

hence $a = b$.

---

[1] A well ordered set is a set X with a binary relation $\leq$ on it such that the following hold for all $x, y, z \in X$:

(1) $x \leq x$. (reflexivity)
(2) If $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetry)
(3) If $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)
(4) Either $x \leq y$ or $y \leq x$. (comparability)
(5) Every nonempty subset of X has a least element.

Recall that **every set can be well ordered**. This is a result in set theory first discovered by Georg Cantor in 1883, and proved by Ernst Zermelo in 1908.

(Q4.) If $\lim\limits_{n\to\infty} a_n = a$, and $\lim\limits_{n\to\infty} b_n = b$, show $\lim\limits_{n\to\infty} a_n b_n = ab$.
Solution: Assume $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers; $a, b \in \mathbb{R}$; and $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$.
We need to prove

$$\forall \epsilon > 0 \qquad \exists n_0 \in \mathbb{N} \qquad \forall_{\substack{n\in\mathbb{N}\\ n\geq n_0}} : \qquad\qquad |a_n b_n - ab| < \epsilon.$$

Pick $\epsilon > 0$. Then, since $\lim\limits_{n\to\infty} a_n = a$,

$$\exists n_1 \in \mathbb{N} \qquad \forall_{\substack{n\in\mathbb{N}\\ n\geq n_1}} : \qquad\qquad |a_n - a| < \frac{\epsilon}{2(|b|+1)}.$$

Furthermore, since every convergent sequence of real numbers is bounded, we have

$$\exists M \in (0,\infty) \qquad \forall n \in \mathbb{N} : \qquad\qquad |a_n| \leq M.$$

Since $\lim\limits_{n\to\infty} b_n = b$,

$$\exists n_2 \in \mathbb{N} \qquad \forall_{\substack{n\in\mathbb{N}\\ n\geq n_2}} : \qquad\qquad |b_n - b| < \frac{\epsilon}{2M}.$$

Take $n_0 = \text{maximum}\,\{n_1, n_2\}$. Then for all integers $n \geq n_0$ we have

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\
&= |a_n(b_n - b) + (a_n - a)b| \\
&\leq |a_n(b_n - b)| + |(a_n - a)b| \\
&= |a_n|\,|b_n - b| + |b|\,|a_n - a| \\
&\leq M\,|b_n - b| + |b|\,|a_n - a| \\
&< M\frac{\epsilon}{2M} + |b|\,\frac{\epsilon}{2(|b|+1)} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.
\end{aligned}$$

This shows $\lim\limits_{n\to\infty} a_n b_n = ab$.