Maths 1124 – Assignment 4

(Q1.) Consider the transformation of $\mathbb{R} \times \mathbb{R}$ onto itself given by $(u, v) \mapsto (x, y)$, where $x = au + bv$, $y = cu + dv$. Find conditions on (a, b, c, d) for this transformation to be bijective.

Solution: The transformation is given by the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\mathbb{R}^2 \to \mathbb{R}^2, \qquad \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au + bv \\ cu + dv \end{bmatrix}.$$

It is bijective $\Leftrightarrow \det(A) \neq 0$ $(ad - bc \neq 0)$.

(Q2.) Prove that the set of all intervals (of length > 0), with both end points rational, is countable.

Solution: The set of all open intervals $a < x < b$ with end points $a, b \in \mathbb{Q}$ is countable because any such interval can be identified with the ordered pair (a, b) consisting of its rational end points a and b (and we know that the set $\mathbb{Q} \times \mathbb{Q}$ is countable).

The set of all closed intervals $a \leq x \leq b$ with both end points rational is also countable because (again) any such interval can be identified with the ordered pair (a, b).

The same idea applies to half-open intervals $a < x \leq b$ and $a \leq x < b$.

Thus, the set of all intervals (of length > 0), with both end points rational, is countable because it is the union of the four countable sets.

(Q3.) Prove that every family of disjoint intervals (of length > 0) is countable.

Solution: This is true because any interval (of length > 0) contains a rational number. Briefly: if $\mathcal{I} = \{I_\lambda : \lambda \in \Lambda\}$ is a (nonempty) family of disjoint intervals (of length > 0) contained in $\mathbb{R}$, then the axiom of choice tells us that it is possible to make a selection of exactly one point, say q_λ, from each I_λ. We can assume $q_\lambda \in \mathbb{Q}$ because between any two distinct real numbers there is always a rational number (by the the Archimedean property of $\mathbb{R}$). Now consider

$$f\colon \Lambda \to \mathbb{Q}, \qquad f(\lambda) = q_\lambda \text{ for all } \lambda \in \Lambda.$$

The function f is injective; therefore $f\colon \Lambda \to f(\Lambda)$ is bijective, and $|\Lambda| = |f(\Lambda)|$. (Here $|\Lambda|$ denotes the cardinal number of Λ or the "number of elements" in Λ.) But $f(\Lambda)$ is a subset of $\mathbb{Q}$, thus $|f(\Lambda)| \leq |\mathbb{Q}|$, i.e., $f(\Lambda)$ is (at most) countable. Hence Λ is countable, and $\mathcal{I}$ is countable (because $|\mathcal{I}| = |\Lambda|$).

(Q4.) Let $f\colon X \to Z$. If $Y \subseteq X$ and f is 1 - 1 prove $f(X \setminus Y) = f(X) \setminus f(Y)$. Is 1 - 1 necessary?

Solution: In order to prove $f(X \setminus Y) = f(X) \setminus f(Y)$, we shall show that each set is a subset of the other.

It is straightforward to show that $f(X \setminus Y) \supseteq f(X) \setminus f(Y)$ (and for this you don't need to assume f is 1 - 1). Indeed: if $z \in f(X) \setminus f(Y)$, then $z \in f(X)$ and $z \notin f(Y)$. Therefore $\exists x \in X \setminus Y$ with $z = f(x)$, and so $z \in f(X \setminus Y)$.

To show the other inclusion, assume f is 1 - 1, and pick $z \in f(X \setminus Y)$. Then there exists one and only one element $x \in X \setminus Y$ with $z = f(x)$. Therefore $z \in f(X)$ and $z \notin f(Y)$, i.e., $z \in f(X) \setminus f(Y)$.

To prove $f(X \setminus Y) = f(X) \setminus f(Y)$, it is necessary for f to be 1 - 1. Here is an example: take $X = \{-1, 0, 1\}$, $Y = \{-1\}$, $Z = \{0, 1\}$, and consider $f\colon X \to Y$ defined by $f(n) = n^2$. Then $f(X \setminus Y) = \{0, 1\} \neq \{0\} = f(X) \setminus f(Y)$, but f is not 1 - 1.