Maths 1124 – Assignment 3

(Q1.) If X and Y are sets, define $X \sim Y$ means X and Y have the same cardinal number, i.e.

$$X \sim Y \;\Leftrightarrow\; \exists f\colon X \to Y \text{ such that } f \text{ is a 1 - 1 and onto function.}$$

Show $\sim$ is an equivalence relation.

Solution: If X, Y, Z are sets, then we have:

- **Reflexivity** $X \sim X$ by the identity function $\mathrm{id}\colon X \to X$, $\mathrm{id}(x) = x$.
- **Symmetry** If $X \sim Y$, there is a 1 - 1 and onto function $f\colon X \to Y$. Then the inverse function $f^{-1}\colon Y \to X$ is 1 - 1 and onto, so $Y \sim X$.
- **Transitivity** If $X \sim Y$ and $Y \sim Z$, there are 1 - 1 and onto functions $f\colon X \to Y$ and $g\colon Y \to Z$. Then the composition of f and g, $g \circ f\colon X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ (for all $x \in X$), is 1 - 1 and onto, so $X \sim Z$.

(Q2.) Show $\exists f\colon X \to X$ which is 1 - 1 but not onto $\Leftrightarrow$ $\exists f\colon X \to X$ which is onto but not 1 - 1.

Solution: Assume X is an infinite set (because any 1 - 1 function from a finite set to itself is automatically onto). We need to show

$$\Big(\exists f\colon X \to X \text{ which is 1 - 1 but not onto}\Big) \Leftrightarrow \Big(\exists g\colon X \to X \text{ which is onto but not 1 - 1}\Big).$$

For the "left to right" implication assume $f\colon X \to X$ is a 1 - 1 but not onto function. Define $g\colon X \to X$ by

$$g(y) = \begin{cases} x & \text{if } y = f(x) \text{ for some } x \in X; \\ y & \text{if } y \in X \setminus \{f(x) : x \in X\}. \end{cases}$$

Then g is onto but not 1 - 1.

For the "right to left" implication assume $g\colon X \to X$ is an onto but not 1 - 1 function. Then X is the union of pairwise disjoint non-empty subsets of X:

$$X = \bigcup_{y \in X} g^{-1}(y)$$

where $g^{-1}(y) = \{x \in X : g(x) = y\}$ is the preimage of y ($y \in X$). For each $y \in X$, choose an element (exactly one) from the set $g^{-1}(y)$; denote it by $c_g(y)$. In fact, this process sets up a new "choice function" which "chooses" exactly one element, $c_g(y)$, from each set $g^{-1}(y)$. Then $f\colon X \to X$ defined by $f(y) = c_g(y)$ (for all $y \in X$) is 1 - 1 but not onto.

(Q3.) Use induction to show that if A_i are countable then $A_1 \times \cdots \times A_n$ is countable for any n.

Solution: Let $n \in \{1, 2, 3, \dots\}$ be a natural number and let $A_1, \dots, A_n$ be countable sets. We shall prove by induction on n that the set $A_1 \times \cdots \times A_n$ is countable.

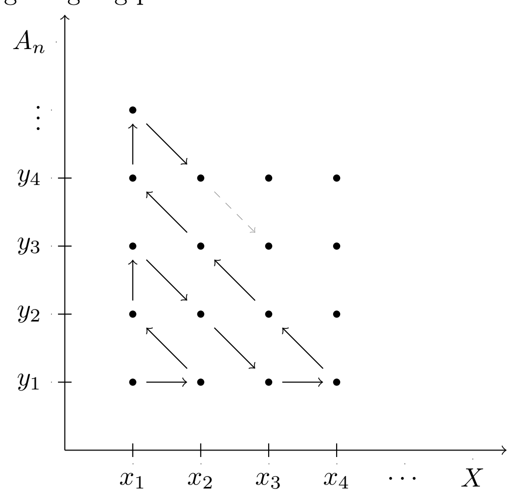
The base case is $n = 1$, and is true since A_1 is countable. Now suppose $n \geq 2$ and the set $X = (A_1 \times \cdots \times A_{n-1})$ is countable. We can list its elements:

$$X = \{x_1, x_2, x_3, \dots\}.$$

Sincee A_n is countable, we can also write:

$$A_n = \{y_1, y_2, y_3, \dots\}.$$

Then $A_1 \times \cdots \times A_n = (A_1 \times \cdots \times A_{n-1}) \times A_n = X \times A_n$ (recall $X \times A_n$ means the set of ordered pairs (x, y) with $x \in X$, $y \in A_n$), and we can list the elements of $X \times A_n$ by following a zig-zag pattern:



$$X \times A_n = \{(x_1, y_1), (x_2, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_2), (x_3, y_1), (x_4, y_1), (x_3, y_2), \dots\}.$$

In other words, we have a pairing

$$\begin{array}{ccccccccc} (x_1, y_1) & (x_2, y_1) & (x_1, y_2) & (x_1, y_3) & (x_2, y_2) & (x_3, y_1) & (x_4, y_1) & (x_3, y_2) & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \end{array}$$

of each element of $X \times A_n$ with a natural number, and so $X \times A_n = A_1 \times \cdots \times A_n$ is countable.

(Q4.) Let $\mathbb{N} = \{0, 1, 2, \dots\}$. Define $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(k, n) = 2^k(2n + 1) - 1$. Show f is 1 - 1 and onto.

Solution: To show f is 1 - 1, we need to show:

$$\forall k, n, l, m \in \mathbb{N}: \qquad f(k, n) = f(l, m) \quad \Rightarrow \quad (k, n) = (l, m).$$

Fix $k, n, l, m \in \mathbb{N}$ and assume $f(k, n) = f(l, m)$. Then $2^k(2n+1)-1 = 2^l(2m+1)-1$, $2^k(2n + 1) = 2^l(2m + 1)$. Note that $k < l$ would imply:

$$\begin{aligned} 2n + 1 &= \frac{2^l}{2^k}(2m + 1), \\ \underbrace{2n + 1}_{odd} &= \underbrace{2^{l-k}(2m + 1)}_{even}, \end{aligned}$$

which is a contradiction; so the case $k < l$ cannot happen. A similar argument shows that we can't have $k > l$. Thus we must have $k = l$. It now follows that: $2^k(2n+1) = 2^k(2m+1)$, $2n+1 = 2m+1$, $n = m$. Hence $(k, n) = (l, m)$; f is 1 - 1.

To show f is onto, we need to show:

$$\forall w \in \mathbb{N} \quad \exists k, n \in \mathbb{N} : \qquad f(k, n) = w.$$

Fix $w \in \mathbb{N}$.

Case 1: If w is even, say $w = 2u$ for some $u \in \{0, 1, 2, \dots\}$, then

$$\begin{aligned} f(0, u) &= 2^0(2u + 1) - 1 \\ &= 2u \\ &= w. \end{aligned}$$

Case 2: Assume w is odd, $w = 2u - 1$ for some $u \in \{1, 2, 3, \dots\}$. Then[1] $u = 2^a(2b + 1)$ for some nonnegative integers $a, b \in \{0, 1, 2, \dots\}$, and

$$\begin{aligned} f(a + 1, b) &= 2^{a+1}(2b + 1) - 1 \\ &= 2 \cdot 2^a(2b + 1) - 1 \\ &= 2u - 1 \\ &= w. \end{aligned}$$

In both cases, there are k and n with $f(k, n) = w$. Thus f is onto.

[1]Every natural number $u \geq 1$ can be expressed as the product of a power of 2 and an odd number, i.e., $\forall u \in \{1, 2, 3, \dots\} \;\; \exists a, b \in \{0, 1, 2, \dots\}$ such that $u = 2^a(2b + 1)$.