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Maths 1124 – Assignment 2

Let $(a_1, a_2, \dots)$ be a bounded sequence of real numbers. Define:

$$\begin{aligned} l_1 &= \sup\{a_1, a_2, \dots\} \ \ (l_1 \text{ is the least upper bound of the set } \{a_1, a_2, \dots\}), \\ g_1 &= \inf\{a_1, a_2, \dots\} \ \ (g_1 \text{ is the greatest lower bound of the set } \{a_1, a_2, \dots\}). \end{aligned}$$

Then $g_1 \leq l_1$ and $[g_1, l_1]$ is the smallest interval such that $\{a_1, a_2, \dots\} \subseteq [g_1, l_1]$. In fact, if $\mathcal{S}$ denotes the set of limits of all possible convergent subsequences of $(a_1, a_2, \dots)$, then $\mathcal{S}$ is a nonempty subset of $[g_1, l_1]$, so both $\inf \mathcal{S}$ and $\sup \mathcal{S}$ belong to $[g_1, l_1]$. Similarly, we can define:

$$\begin{aligned} l_2 &= \sup\{a_2, a_3, \dots\} \ \ (l_2 \text{ is the least upper bound of the set } \{a_2, a_3, \dots\}), \\ g_2 &= \inf\{a_2, a_3, \dots\} \ \ (g_2 \text{ is the greatest lower bound of the set } \{a_2, a_3, \dots\}). \end{aligned}$$

Then $g_2 \leq l_2$ and $[g_2, l_2]$ is the smallest interval such that $\{a_2, a_3, \dots\} \subseteq [g_2, l_2]$. But since $\{a_2, a_3, \dots\} \subseteq \{a_1, a_2, \dots\} \subseteq [g_1, l_1]$, we must have $[g_2, l_2] \subseteq [g_1, l_1]$ (because $[g_2, l_2]$ is the smallest interval containing $\{a_2, a_3, \dots\}$ and $[g_1, l_1]$ is an interval containing $\{a_2, a_3, \dots\}$).

In general, define:

$$\begin{aligned} l_n &= \sup\{a_n, a_{n+1}, \dots\}, \\ g_n &= \inf\{a_n, a_{n+1}, \dots\}. \end{aligned}$$

Then $g_n \leq l_n$, $[g_n, l_n]$ is the smallest interval such that $\{a_n, a_{n+1}, \dots\} \subseteq [g_n, l_n]$, and we have

$$[g_n, l_n] \subseteq \cdots \subseteq [g_2, l_2] \subseteq [g_1, l_1]$$

or, equivalently,

$$g_1 \leq g_2 \leq \cdots \leq g_n \leq l_n \leq \cdots \leq l_2 \leq l_1.$$

The (infinite) sequence $(g_1, g_2, \dots)$ is increasing and bounded above, so it must converge. Therefore $\lim\limits_{n\to\infty} g_n = g$ for some real number g.
The sequence $(l_1, l_2, \dots)$ must also converge because it is decreasing and bounded below. Thus $\lim\limits_{n\to\infty} l_n = l$ for some real number l.
Since $g_n \leq l_n$ for all n, we must have $g \leq l$; g and l are called the *upper* and *lower limit* of $(a_1, a_2, \dots)$, respectively; we use the notation: $\overline{\lim}\ a_n = \limsup a_n = l$, $\underline{\lim}\ a_n = \liminf a_n = g$.

We have shown that if $(a_1, a_2, \dots)$ is a bounded sequence of real numbers, then $\underline{\lim}\ a_n$ and $\overline{\lim}\ a_n$ exist and:

$$\begin{aligned} \overline{\lim}\ a_n = \limsup a_n &= \lim \Big(\sup\{a_1, a_2, \dots\}, \sup\{a_2, a_3, \dots\}, \dots \Big) \\ &= \inf\{\sup\{a_1, a_2, \dots\}, \sup\{a_2, a_3, \dots\}, \dots\}, \end{aligned}$$

$$\begin{aligned}\underline{\lim}\ a_n \ = \ \liminf a_n \ &= \ \lim\Big(\inf\{a_1,a_2,\dots\}, \inf\{a_2,a_3,\dots\},\dots\Big)\\ &= \ \sup\{\inf\{a_1,a_2,\dots\}, \inf\{a_2,a_3,\dots\},\dots\}.\end{aligned}$$

Letting $\mathcal{S}$ denote the set of limits of all possible convergent subsequences of $(a_1, a_2, \dots)$, we have:

$$\begin{aligned}\overline{\lim}\ a_n \ &= \ \limsup a_n \ &= \ \sup\ \mathcal{S},\\ \underline{\lim}\ a_n \ &= \ \liminf a_n \ &= \ \inf\ \mathcal{S}.\end{aligned}$$

(Q1.) Prove that if $\lim_{n\to\infty} a_n = A$, then

$$\overline{\lim}\ a_n \ = \ \underline{\lim}\ a_n \ = \ A.$$

Solution. If $\lim_{n\to\infty} a_n = A$, then every subsequence of $(a_1, a_2, \dots)$ converges to the same limit A. Therefore $\mathcal{S}$, the set of limits of all possible convergent subsequences of $(a_1, a_2, \dots)$, has exactly one point, A, so

$$\begin{aligned}\overline{\lim}\ a_n \ &= \ \sup\ \mathcal{S} \ = \ \sup\ \{A\} \ = \ A,\\ \underline{\lim}\ a_n \ &= \ \inf\ \mathcal{S} \ = \ \inf\ \{A\} \ = \ A.\end{aligned}$$

(Q2.) We have: $\limsup\{-a_n\} \ = \ -\liminf\{a_n\}$.

Also:

$$\begin{aligned}\limsup\{a_n+b_n\} \ &\le \ \limsup\{a_n\} + \limsup\{b_n\},\\ \liminf\{a_n+b_n\} \ &\ge \ \liminf\{a_n\} + \liminf\{b_n\},\end{aligned}$$

whenever the right sides of the inequalities are defined (i.e., not $\mp\infty \pm \infty$).

Similarly, if $a_n \ge 0$, $b_n \ge 0$ are bounded, then:

$$\begin{aligned}\limsup\{a_n \cdot b_n\} \ &\le \ (\limsup\{a_n\}) \cdot (\limsup\{b_n\}),\\ \liminf\{a_n \cdot b_n\} \ &\ge \ (\liminf\{a_n\}) \cdot (\liminf\{b_n\}),\end{aligned}$$

whenever the right sides of the inequalities are defined (i.e., not $0 \cdot \infty$).

If $\lim b_n$ exists, then we have:

$$\begin{aligned}\limsup\{a_n+b_n\} \ &= \ \limsup\{a_n\} + \lim b_n,\\ \liminf\{a_n+b_n\} \ &= \ \liminf\{a_n\} + \lim b_n,\\ \limsup\{a_n \cdot b_n\} \ &= \ (\limsup\{a_n\}) \cdot (\lim b_n),\\ \liminf\{a_n \cdot b_n\} \ &= \ (\liminf\{a_n\}) \cdot (\lim b_n).\end{aligned}$$

(Q3.) If $\limsup\{a_n\} = L$, prove that $\forall \epsilon > 0$, $\exists$ infinitely many a_n such that $a_n > L - \epsilon$.
Solution. Since $\limsup\{a_n\} = L$, there is an infinite subsequence (a_{n_k}) of (a_n) with $\lim_{k\to\infty} a_{n_k} = L$. So every neighbourhood of the point L contains all but (possibly) a finite number of the terms a_{n_k}, i.e.,

$$\forall\epsilon > 0 \qquad \exists k_0 \in \mathbb{N} \qquad \forall k \in \mathbb{N}, k \ge k_0 \qquad \text{we have:} \qquad L - \epsilon < a_{n_k} < L + \epsilon.$$

In particular, $\forall\epsilon > 0$, $\exists$ infinitely many a_{n_k} such that $a_{n_k} > L - \epsilon$.