Maths 1124 – Assignment 1

(Q1.) Truth Tables: Whenever we encounter a complex formula like $(P \wedge Q)'$, we work from the inside out, just as we might do if we had to evaluate an arithmetic expression, like $-(m \cdot n)$. Thus, we start with the P and Q columns, then construct the $P \wedge Q$ column, and finally, the $(P \wedge Q)'$ column. Truth tables can be used to prove logical equivalences, like $(P \wedge Q)' \equiv P' \vee Q'$. To prove this, we construct a truth table with columns for both $(P \wedge Q)'$ and $P' \vee Q'$:

$$(P \wedge Q)' \equiv P' \vee Q'$$

P	Q	$P \wedge Q$	$(P \wedge Q)'$	P'	Q'	$P' \vee Q'$
1	1	1	**0**	0	0	**0**
1	0	0	**1**	0	1	**1**
0	1	0	**1**	1	0	**1**
0	0	0	**1**	1	1	**1**

Since the $(P \wedge Q)'$ and $P' \vee Q'$ columns contain the same truth values in all rows (for all possible truth values of the variables involved), we have $(P \wedge Q)' \equiv P' \vee Q'$.

More Examples:

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
1	1	1	1	**1**	1	1	**1**
1	1	0	0	**1**	1	1	**1**
1	0	1	0	**1**	1	1	**1**
1	0	0	0	**1**	1	1	**1**
0	1	1	1	**1**	1	1	**1**
0	1	0	0	**0**	1	0	**0**
0	0	1	0	**0**	0	1	**0**
0	0	0	0	**0**	0	0	**0**

$$(P \vee Q) \Rightarrow R \equiv (P \Rightarrow R) \wedge (Q \Rightarrow R)$$

P	Q	R	$P \vee Q$	$(P \vee Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
1	1	1	1	**1**	1	1	**1**
1	1	0	1	**0**	0	0	**0**
1	0	1	1	**1**	1	1	**1**
1	0	0	1	**0**	0	1	**0**
0	1	1	1	**1**	1	1	**1**
0	1	0	1	**0**	1	0	**0**
0	0	1	0	**1**	1	1	**1**
0	0	0	0	**1**	1	1	**1**

The equivalence $(P \vee Q) \Rightarrow R \;\equiv\; (P \Rightarrow R) \wedge (Q \Rightarrow R)$ can also be proved as follows:

$$\begin{aligned}
(P \vee Q) \Rightarrow R &\equiv (P \vee Q)' \vee R \\
&\equiv (P' \wedge Q') \vee R \\
&\equiv (P' \vee R) \wedge (Q' \vee R) \\
&\equiv (P \Rightarrow R) \wedge (Q \Rightarrow R).
\end{aligned}$$

(Q2.) Write out the negation for each statement:

(a) Some footballers are short.
  Negation: All footballers are tall.
(b) All the lights are on.
  Negation: Some light is off.
(c) No bounded interval contains infinitely many integers.
  Negation: There is a bounded interval containing infinitely many integers.
(d) $\exists\, x$ in S such that $x \geq 5$.
  Negation: $\forall\, x$ in S, $x < 5$.
(e) $\forall\, x$ such that $0 < x < 1$, $f(x) < 2$ or $f(x) > 5$.
  Negation: $\exists\, x_0$ in $(0, 1)$ such that $f(x_0) \geq 2$ and $f(x_0) \leq 5$ (this can also be writen as: $\exists x_0 \in (0, 1)$ such that $f(x_0) \in [2, 5]$).
(f) If $x > 5$, then $\exists y > 0$ such that $x^2 > 25 + y$. This statement can be written as:

$$x > 5 \Rightarrow \Big(\exists y > 0 \;\text{ such that }\; x^2 > 25 + y\Big).$$

Since $p \to q \;\equiv\; (p') \vee q$, the above is equivalent to:

$$x \leq 5 \;\text{ or }\; \Big(\exists y > 0 \;\text{ such that }\; x^2 > 25 + y\Big).$$

Since $(p \vee q)' \;\equiv\; (p') \wedge (q')$, the negation of the above statement (which is the negation of the original statement) is:
  Negation: $x > 5$, and $\forall y > 0$ we have $x^2 \leq 25 + y$.

(Q3.) If $A_1, A_2, \ldots, A_n, \ldots$ is a sequence of sets, then the limits inferior and superior are defined to be:

$$\begin{aligned}
\liminf A_n &= \bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} A_m\right) \\
&= (A_1 \cap A_2 \cap \cdots) \cup (A_2 \cap A_3 \cap \cdots) \cup (A_3 \cap A_4 \cap \cdots) \cup \cdots \\
&= \{x \,:\, x \in A_n \;\text{ for all but finitely many } n\}
\end{aligned}$$

and

$$\begin{aligned}
\limsup A_n &= \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{\infty} A_m\right) \\
&= (A_1 \cup A_2 \cup \cdots) \cap (A_2 \cup A_3 \cup \cdots) \cap (A_3 \cup A_4 \cup \cdots) \cap \cdots \\
&= \{x : x \in A_n \;\text{ for infinitely many } n\}.
\end{aligned}$$

(a) If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, then

$$\begin{aligned}
\liminf A_n &= \bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} A_m\right) \\
&= (A_1 \cap A_2 \cap \cdots) \cup (A_2 \cap A_3 \cap \cdots) \cup (A_3 \cap A_4 \cap \cdots) \cup \cdots \\
&= A_1 \cup A_2 \cup A_3 \cup \cdots \\
&= \bigcup_{n=1}^{\infty} A_n
\end{aligned}$$

and

$$\begin{aligned}
\limsup A_n &= \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{\infty} A_m\right) \\
&= (A_1 \cup A_2 \cup \cdots) \cap (A_2 \cup A_3 \cup \cdots) \cap (A_3 \cup A_4 \cup \cdots) \cap \cdots \\
&= \left(\bigcup_{n=1}^{\infty} A_n\right) \cap \left(\bigcup_{n=1}^{\infty} A_n\right) \cap \left(\bigcup_{n=1}^{\infty} A_n\right) \cap \cdots \\
&= \bigcup_{n=1}^{\infty} A_n.
\end{aligned}$$

(b) If $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, then

$$\begin{aligned}
\liminf A_n &= \bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} A_m\right) \\
&= (A_1 \cap A_2 \cap \cdots) \cup (A_2 \cap A_3 \cap \cdots) \cup (A_3 \cap A_4 \cap \cdots) \cup \cdots \\
&= \left(\bigcap_{n=1}^{\infty} A_n\right) \cup \left(\bigcap_{n=1}^{\infty} A_n\right) \cup \left(\bigcap_{n=1}^{\infty} A_n\right) \cup \cdots \\
&= \bigcap_{n=1}^{\infty} A_n
\end{aligned}$$

and

$$\begin{aligned}
\limsup A_n &= \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{\infty} A_m\right) \\
&= (A_1 \cup A_2 \cup \cdots) \cap (A_2 \cup A_3 \cup \cdots) \cap (A_3 \cup A_4 \cup \cdots) \cap \cdots \\
&= A_1 \cap A_2 \cap A_3 \cap \cdots \\
&= \bigcap_{n=1}^{\infty} A_n.
\end{aligned}$$

Example: Consider $A_n = [0, n]$ if n is even; $A_n = [0, \frac{1}{n}]$ if n is odd. Then $\liminf A_n = \{0\}$, $\limsup A_n = [0, \infty)$.