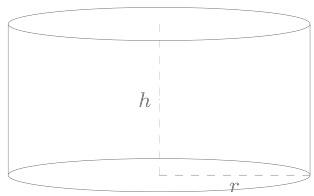
23 November 2011

30. A closed cylindrical can is to have a surface area of S square units. Show that the can of maximum volume is achieved when the height is equal to the diameter.

Solution: Let h be the height of the can, r the radius. Let us agree to measure h and r in fixed units u (for example $u =$ cm, or $u =$ m, ...). Set $d = 2r$ (diameter).



The surface area of the can is $S = 2\pi r^2 + 2\pi rh$, and the volume of the can is $V = \pi r^2 h$. From the "surface" formula $h = \frac{S-2\pi r^2}{2\pi r}$, so we can express the volume V as a function of one variable r, namely $V = \pi r^2 \frac{S-2\pi r^2}{2\pi r} = \frac{Sr-2\pi r^3}{2} = \frac{S}{2}r - \pi r^3$. Now find a maximum of the continuous function:

$$V\colon (0,\infty) \to \mathbb{R}, \qquad V(r) = \frac{S}{2}r - \pi r^3.$$

The derivative is: $V'(r) = \frac{S}{2} - 3\pi r^2$, and $V'(r) = 0$ if and only if $\frac{S}{2} = 3\pi r^2$, or

$$S = 6\pi r^2.$$

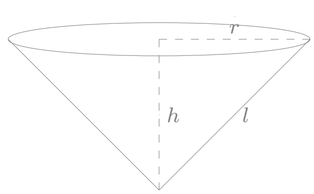
The second derivative of V is $V'' = -6\pi r$, and since $-6\pi\sqrt{\frac{S}{6\pi}} < 0$, the function V has a (local) maximum at $r = \sqrt{\frac{S}{6\pi}}$. We have

$$2\pi r^2 + 2\pi rh = 6\pi r^2,$$

hence $h = 2r = d$. So the can of maximum volume is achieved when the height is equal to the diameter.

37. A cone-shaped paper drinking cup is to hold 100 cm^3 of water. Find the height and radius of the cup that will require the least amount of paper.

Solution: Let h be the height of the cone, let r be the radius of the circle at the top of the cone, and let l the lateral height of the cone.



The volume of the cone is $V = \frac{1}{3}\pi r^2 h$, so $\frac{1}{3}\pi r^2 h = 100$, $\pi r^2 h = 300$, $h = \frac{300}{\pi r^2}$. Since $l = \sqrt{r^2 + h^2} = \sqrt{r^2 + (\frac{300}{\pi r^2})^2}$, the lateral surface area of the cone is $A = \pi r l = \pi r\sqrt{r^2 + (\frac{300}{\pi r^2})^2} = \sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}$. We now have the area of the paper used in the cone as a function of r, namely

$$A\colon (0,\infty) \to \mathbb{R}, \qquad A(r) = \sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}.$$

The derivative is: $A'(r) = \frac{1}{2\sqrt{\pi^2 r^4 + \frac{300^2}{r^2}}}(4\pi^2 r^3 - \frac{2\cdot 300^2}{r^3})$. Find r_0 such that $A'(r_0) = 0$. Then the height and radius of the cup that will require the least amount of paper are respectively $h = \frac{300}{\pi (r_0)^2}$ and r_0.

40. A commercial cattle ranch currently allows 20 steers per acre of grazing land. On average the steers weigh 2000 lb at market. Estimates by the Department of Agriculture indicate that the average market weight per steer will be reduced by 50 lb for each additional steer added per acre of grazing land. How many steers per acre should the ranch allow in order for the ranch to get the largest possible total market weight for its cattle?

Solution: Currently the ranch has 20 steer/acre; the steer weigh 2000 lb each ($20 \cdot 2000 = 40000$ lb). Suppose we add x more steer per acre, $20 + x$ steer/acre. Their average weight will drop to $2000 - 50x$ lb each. So the total weight will be $w = (20 + x)(2000 - 50x)$ lb. The function we want to maximize is $w(x) = 40000 + 1000x - 50x^2$; $w'(x) = 0$ if and only if $x = 10$. Thus the ranch should allow 30 steers/acre in order to get the largest possible total market weight for its cattle.

41. A company mines low-grade nickel ore. If the company mines x tones of ore, it can sell the ore for $p = 225 - 0.25x$ dolars per ton. Find revenue and marginal revenue functions. At what level of production would the company obtain maximum revenue?

Solution: Revenue function: Suppose x denotes the number of units a company plan to produce or sell; a revenue function $R(x)$ is set up as follows: $R(x) =$ (price per unit) $\cdot$ (number of units produced or sold), so $R(x) = p(x)x$; a marginal revenue is $R'(x)$. We have $p(x) = 225 - 0.25x$; Revenue function: $R(x) = p(x)x = 225x - 0.25x^2$; marginal revenue function: $R'(x) = 225 - 0.5x$. When revenue is maximum we have $R'(x) = 0$, so $225 = 0.5x$, $x = 450$. Therefore, if 450 tons are producd, then revenue is maximum.

44. A firm determines that x units of its product can be sold daily at p dollars per unit, where $x = 1000 - p$. The cost of producing x units per day is $C(x) = 3000 + 2x$.

(a) Find the revenue function $R(x)$.

(b) Find the profit function $P(x)$.

(c) Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit.

(d) Find the maximum profit.

(e)What price per unit must be charged to obtain the maximum profit?

Solution: (a) $R(x) = px = p(1000 - p)$.

(b) A profit function P gives the total profit $P(x)$ from the sale of x items. The profit, cost and revenue functions are related by the formula $P(x) = R(x) - C(x)$ (Revenue - Cost). We have $P = p(1000 - p) - (3000 + 2(1000 - p))$.
(c) Assuming that the production capacity is at most 500 units per day, which means $0 \leq x \leq 500$, determine how many units the company must produce and sell each day to maximize the profit. So first find $\frac{dP}{dp}$, then solve $\frac{dP}{dp} = 0$ for p - call it p_0. Use this to find an x.
(d) The maximum profit is $P(p_0)$.
(e) What price per unit must be charged to obtain the maximum profit? Since $x = 1000 - p$, $p = 1000 - x$.