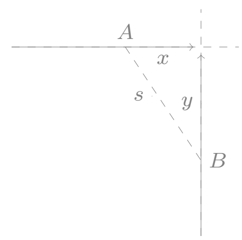
14. Two commercial airplanes are flying at 40,000 ft along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when plane A is 5 nautical miles from the intersection point and plane B is 12 nautical miles from the intersection point?
Solution: Let x be the distance A is from the intersection point, and let y be the distance B is from the intersection point. Let s be the distance between A and B, so $s^2 = x^2 + y^2$ (see the picture below).



Note that s, x and y are functions of time, t, so to emphasize this we should write

$$s(t)^2 = x(t)^2 + y(t)^2.$$

Differentiate both sides of the above equation, with respect to t, to get

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Dividing by 2 gives $s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$, and dividing by s gives

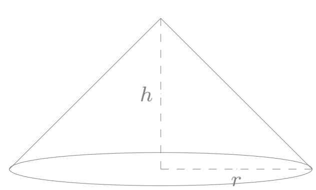
$$\frac{ds}{dt} = \frac{1}{s}(x\frac{dx}{dt} + y\frac{dy}{dt}).$$

Since $\frac{dx}{dt} = -442$, $\frac{dy}{dt} = -481$, $x = 5$ nautical miles, and $y = 12$ nautical miles, we have

$$\begin{aligned}\frac{ds}{dt} &= \frac{1}{s}(x\frac{dx}{dt} + y\frac{dy}{dt}) \\ &= \frac{1}{\sqrt{5^2+12^2}}(5(-442) + 12(-481)) \\ &= \frac{-7982}{13} = -614.\end{aligned}$$

Thus the distance between the planes is decreasing at a rate of 614 knots.

17. Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always $\frac{3}{8}$ (three-eighths) of the base diameter. How fast are the (a) height and (b) radius changing when the pile is 4 m high?
Solution: The pile of sand will be (approximately) in the shape of a circular cone of base-radius r, height h, and volume $V = \frac{1}{3}\pi(r^2)h$ (see the picture below).



The height h of the pile is always $\frac{3}{8}$ (three-eighths) of the base diameter, so $h = \frac{3}{8}(r+r) = \frac{3r}{4}$, or equivalently $r = \frac{4h}{3}$. Substituting $r = \frac{4h}{3}$ into the volume expression we get

$$V = \frac{1}{3}\pi(\frac{4h}{3})^2 h = \frac{16\pi}{27}h^3.$$

Note that the volume V and the height h are both functions of time, t, so to emphasize this we shall write

$$V(t) = \frac{16\pi}{27}h(t)^3.$$

Differentiate both sides of the above equation, with respect to t, to get

$$\frac{dV}{dt} = \frac{16\pi}{9}h^2\frac{dh}{dt}. \qquad (*)$$

We know that the sand falls from a conveyor belt at the rate of 10 m^3/min (10 m^3 per minute), so $\frac{dV}{dt} = 10$ m^3/min. Hence:

(a) Substituting $\frac{dV}{dt} = 10$ and $h = 4$ into equation (*) yields $10 = \frac{16\pi \cdot 4^2}{9}\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{90}{256\pi} \approx$ 11.19 cm/sec.

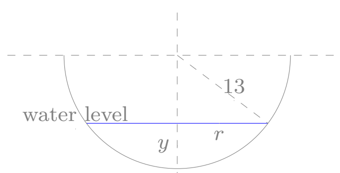
(b) Since $r = \frac{4h}{3}$, we have $\frac{dr}{dt} = \frac{4}{3}\frac{dh}{dt} = \frac{4}{3}\frac{90}{256\pi} = \frac{15}{32\pi} \approx 14.92$ cm/sec.

19. Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m. Given that the volume of water in a hemispherical bowl of radius R is $V = \frac{\pi}{3}y^2(3R - y)$ when the water is y units deep, answer the following questions:

(a) At what rate is the water level changing when the water is 8 m deep?

(b) What is the radius r of the water's surface when the water is y m deep?

(c) At what rate is the radius r changing when the water is 8 m deep?



Solution: Certainly the volume $V = \frac{\pi}{3}y^2(3R - y)$ of water (in a hemispherical bowl of radius R) and the water depth y are functions of time, t, and to emphasize this we can write

$$V(t) = \frac{\pi}{3}y(t)^2(3R - y(t)).$$

$V = V(t)$ is the volume of water in the bowl at time t, so the rate-of-change of V is $\frac{dV}{dt}$.

(a) Differentiate both sides of $V(t) = \frac{\pi}{3}y(t)^2(3R - y(t)) = \pi R y(t)^2 - \frac{\pi}{3}y(t)^3$ with respect to t

$$\begin{aligned}\frac{dV}{dt} &= \pi R \cdot 2y\frac{dy}{dt} - \frac{\pi}{3}3y^2\frac{dy}{dt} \\ &= 2\pi R \cdot y\frac{dy}{dt} - \pi y^2\frac{dy}{dt} \\ &= (2\pi R y - \pi y^2)\frac{dy}{dt}.\end{aligned}$$

Hence, $\frac{dy}{dt} = (2\pi R y - \pi y^2)^{-1}\frac{dV}{dt}$. Substitute into this equation $R = 13$ and $y = 8$ ($\frac{dy}{dt} = \frac{-1}{24\pi}$ m/min).

(b) If the water is y deep then the water-level is $13-y$, thus $r^2+(13-y)^2 = 13^2$, $r^2+(13-y)^2 = 169$, and so $r = \sqrt{26y - y^2}$m.

(c) Differentiating $r = \sqrt{26y - y^2}$ with respect to t gives

$$\frac{dr}{dt} = \frac{1}{2}(26y - y^2)^{-1/2}(26 - 2y)\frac{dy}{dt}, \quad \frac{dr}{dt} = \frac{13 - y}{\sqrt{26y - y^2}}\frac{dy}{dt}, \quad \frac{dr}{dt}|_{(y=8)} = \frac{13 - 8}{\sqrt{26 \cdot 8 - 8^2}}\frac{-1}{24\pi} = \frac{-5}{288\pi} \text{ m/min.}$$