1. Show that there is a root of the equation $4x^3-6x^2+3x-2=0$ between 1 and 2. Estimate it to 2 decimal place.
Solution: Consider the function $f\colon \mathbb{R}\to\mathbb{R}$ defined by $f(x)=4x^3-6x^2+3x-2$. Note: $f(1)=-1$ and $f(2)=12$, and $f(1)<0<f(2)$. The intermediate value theorem implies that there is c between 1 and 2 with $f(c)=0$, i.e. c is a root of the equation $4x^3-6x^2+3x-2=0$. Use a calculator to check $f(1.2)<0<f(1.3)$, and the intermediate value theorem (again) to get a point c between 1.2 and 1.3 with $f(c)=0$. Similarly, since $f(1.22)<0<f(1.23)$, we get $c\in(1.22,\ 1.23)$. Estimation: $c\approx 1.22$.

2. Show that every (real) polynomial of odd order has at least one real root.
Solution: This follows from the intermediate value theorem.

3.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{\tan 3x} &= \lim_{x\to 0}\frac{\sin 3x}{\frac{\sin 3x}{\cos 3x}}\\ &= \lim_{x\to 0}\cos 3x\\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\frac{3x-\sin kx}{x} &= \lim_{x\to 0}(\frac{3x}{x}-\frac{k\sin kx}{kx})\\ &= 3-k\end{aligned}$$

$$\begin{aligned}\lim_{x\to -1}\frac{\sin(x+1)}{x^2-1} &= \lim_{x\to -1}\frac{\sin(x+1)}{(x-1)(x+1)}\\ &= \lim_{x\to -1}\frac{1}{x-1}\frac{\sin(x+1)}{x+1}\\ &= -\frac{1}{2}\end{aligned}$$

(19a) $\lim_{x\to 2}f(x)=5$. (21a) For $\epsilon>0$ take $\delta=\frac{\epsilon}{4}$.

4. Consider the function $f(x)=\frac{\tan kx}{x}$ if $-\frac{\pi}{2k}<x<0$ and $f(x)=\frac{\sqrt{x+k^2}-k}{x}$ if $x>0$. Find the positive value of k for which f is continuous at $x=0$. Find $\lim_{x\to\infty}f(x)$.
Solution: Notice that

$$\lim_{x\to 0}\frac{\tan kx}{x} = \lim_{x\to 0}\frac{k\sin kx}{kx\cos kx} = k,$$

$$\lim_{x\to 0}\frac{\sqrt{x+k^2}-k}{x} = \lim_{x\to 0}\frac{\sqrt{x+k^2}-k}{x}\frac{\sqrt{x+k^2}+k}{\sqrt{x+k^2}+k} = \frac{1}{2k}.$$

To get f to be continuous at 0, take $k=\frac{\sqrt{2}}{2}$ (since $k=\frac{1}{2k}$, $k=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$) and define $f(0)=k$. Then $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\sqrt{x+k^2}-k}{x}=\lim_{x\to\infty}(\frac{\sqrt{x+k^2}-k}{x}\frac{\sqrt{x+k^2}+k}{\sqrt{x+k^2}+k})=0$.