19 October 2011

Exercise 10.

Let $f: \mathbb{R} \to \mathbb{R}$ be a function, and let $a, L \in \mathbb{R}$.
Prove that if $\lim_{x\to a^+} f(x) = L = \lim_{x \to a^-} f(x)$, then $\lim_{x\to a} f(x) = L$.

Proof: Since $\lim_{x\to a^+} f(x) = L$, we have:

$$\forall \varepsilon > 0 \ \exists \delta_1 > 0 \ \forall x \in \mathbb{R} \ \left(0 < x - a < \delta_1\right) \Longrightarrow |f(x) - L| < \varepsilon. \quad (P)$$

Since $\lim_{x\to a^-} f(x) = L$, we have:

$$\forall \varepsilon > 0 \ \exists \delta_2 > 0 \ \forall x \in \mathbb{R} \ \left(0 < a - x < \delta_2\right) \Longrightarrow |f(x) - L| < \varepsilon. \quad (M)$$

To prove $\lim_{x\to a} f(x) = L$ we need to prove that

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in \mathbb{R} \ \ 0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

1. Fix $\varepsilon > 0$. Then by (P) there exists $\delta_1 > 0$ such that
$\forall x \in \mathbb{R} \ \left(0 < x - a < \delta_1\right) \Longrightarrow |f(x) - L| < \varepsilon. \quad (P2)$
By (M) there exists also $\delta_2 > 0$ such that
$\forall x \in \mathbb{R} \ \left(0 < a - x < \delta_2\right) \Longrightarrow |f(x) - L| < \varepsilon. \quad (M2)$

2. Take $\delta = \min\{\delta_1, \delta_2\}$

3. Then for any $x \in \mathbb{R}$ with $0 < |x - a| < \delta$ we have:
$x \neq a$ and $-\delta < x - a < \delta$, $a - \delta < x < a + \delta$.

CASE 1: $0 < x - a < \delta \leq \delta_1$
$0 < x - a < \delta_1$
$\Longrightarrow |f(x) - L| < \varepsilon$ by (P2)

CASE 2: $0 < a - x < \delta \leq \delta_2$
$0 < a - x < \delta_2$
$\Longrightarrow |f(x) - L| < \varepsilon$ by (M2)

It follows that $|f(x) - L| < \varepsilon$.

Exercise 11. Prove
$$\lim_{x \to 2}(x^3 + 1) = 9. \tag{1}$$

We will complete this exercise in two steps. Step one is **Scratch Work** and step two is the **Proof**.

Scratch Work: To prove (1) we need to prove:

$$\forall\, \epsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in \mathbb{R} \qquad [(0 < |x - 2| < \delta) \Longrightarrow |x^3 + 1 - 9| < \epsilon].$$

So given $\epsilon > 0$, we need to find $\delta > 0$ (in terms of ϵ) so that for all real numbers x satisfying $0 < |x - 2| < \delta$ we have $|x^3 - 8| < \epsilon$.

To deduce what δ must be, note that $0 < |x - 2| < \delta$ implies $x \neq 2$ and $2 - \delta < x < 2 + \delta$. We can make a restriction to consider only those δ's that are at most 1, i.e. $\delta \in (0, 1]$. Then x must be greater than 1 but less than 3 ($x \neq 2$), and since the polynomial $x^2 + 2x + 4$ is bounded on the interval $[1, 3]$ by 19 ($19 = 3^2 + 2 \cdot 3 + 4$), we have:

$$\begin{aligned} |x^3 - 8| &= |(x-2)(x^2 + 2x + 4)| \\ &= |x - 2|\,(x^2 + 2x + 4) \\ &< \delta \cdot 19. \end{aligned}$$

Now, $\delta \cdot 19 \leq \epsilon$ if and only if $\delta \leq \frac{\epsilon}{19}$, so we can pick $\delta = \min\left\{1, \frac{\epsilon}{19}\right\}$.

Proof: Fix $\epsilon > 0$ and let $\delta = \min\left\{1, \frac{\epsilon}{19}\right\}$. Then for all real numbers x satisfying $0 < |x - 2| < \delta$ we have: $x \neq 2$, $1 \leq 2 - \delta < x < 2 + \delta \leq 3$, and

$$\begin{aligned} |x^3 - 8| &= |(x-2)(x^2 + 2x + 4)| \\ &= |x - 2|\,(x^2 + 2x + 4) \\ &< \delta \cdot 19 \;\leq\; \epsilon. \end{aligned}$$