

Exercise 6

If \sim is an equivalence relation on X then the set of all equivalence classes form a partition on X ($X \neq \emptyset$).

Proof: If $x \in X$, the equivalence class of x is $[x] = \{y \in X : y \sim x\}$.

The set of all equivalence classes is $X/\sim = \{[x] : x \in X\}$.

Need to prove: ① $\emptyset \notin X/\sim$

$$\text{② } \bigcup_{x \in X} [x] = X$$

③ For all $x, y \in X$: either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

① For any $x \in X$ we have: $x \sim x$, $x \in [x]$, $[x] \neq \emptyset$; hence $\emptyset \notin X/\sim$.

$$\text{② } \text{If } x_0 \in \bigcup_{x \in X} [x]$$

then $x_0 \in [y]$ for some $y \in X$.

But $[y] \subseteq X$, so $x_0 \in X$.

$$\text{If } x_0 \in X$$

then $x_0 \in [x_0]$.

$$\text{Thus } x_0 \in \bigcup_{x \in X} [x].$$

③ Let $x, y \in X$. Assume $[x] \neq [y]$. We will show that $[x] \cap [y] = \emptyset$ by the method of contradiction.

Assume $[x] \cap [y] \neq \emptyset$.

Then there exists $z \in [x] \cap [y]$,
so $z \in [x]$ and $z \in [y]$.

This means that $x \sim z$ and $z \sim y$.

By transitivity of \sim we have $x \sim y$,
so $[x] = [y]$, a contradiction.

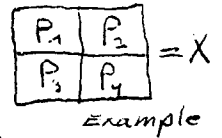
Exercise 1

Let $\{P_\alpha : \alpha \in A\}$ be a partition of X , meaning:

$$\textcircled{1} \forall \alpha \in A \quad P_\alpha \neq \emptyset$$

$$\textcircled{2} \bigcup_{\alpha \in A} P_\alpha = X$$

$$\textcircled{3} \text{ For } \alpha, \beta \in A: \text{ either } P_\alpha = P_\beta \text{ or } P_\alpha \cap P_\beta = \emptyset.$$



For $x, y \in X$ define $x \sim y \iff x, y \in P_\alpha$ for some $\alpha \in A$.

Prove that \sim is an equivalence relation on X .

Proof: Reflexivity: If $x \in X$ then $x \in P_\alpha$ for some $\alpha \in A$. Thus $x \sim x$.

Symmetry: If $x \sim y$ then $x, y \in P_\alpha$ for some $\alpha \in A$.
 Also $y, x \in P_\alpha$ so $y \sim x$.

Transitivity: Assume $x \sim y$ and $y \sim z$.
 Then $x, y \in P_\alpha$ for some $\alpha \in A$
 and $y, z \in P_\beta$ for some $\beta \in A$.
 Now $y \in P_\alpha \cap P_\beta$, so $P_\alpha \cap P_\beta \neq \emptyset$
 and therefore $P_\alpha = P_\beta$.
 Hence $x \in P_\alpha, z \in P_\beta = P_\alpha, x, z \in P_\alpha, x \sim z$.