

14 Dec 2011

1. (26) Let V be the volume of the solid that results when the region enclosed by $y = \frac{1}{x}$, $y = 0$, $x = 2$, and $x = b$ ($0 < b < 2$) is revolved about the x -axis. Find the value of b for which $V = 3$.

Solution: The volume V is given by

$$V = \pi \int_b^2 \left(\left(\frac{1}{x} \right)^2 - 0^2 \right) dx = \pi \int_b^2 \frac{1}{x^2} dx = \pi \left(\frac{1}{b} - \frac{1}{2} \right).$$

Find the value for b which satisfy $V = 3$, i.e. solve the equation $\pi \left(\frac{1}{b} - \frac{1}{2} \right) = 3$ for b . [$\pi \left(\frac{1}{b} - \frac{1}{2} \right) = 3$, $\frac{1}{b} - \frac{1}{2} = \frac{3}{\pi}$, $\frac{1}{b} = \frac{3}{\pi} + \frac{1}{2}$, $b = \frac{2\pi}{6+\pi} \approx 0.7$]

2. (27) Find the volume of the solid generated when the region enclosed by $y = \sqrt{x+1}$, $y = \sqrt{2x}$, and $y = 0$ is revolved about the x -axis. [*Hint:* Split the solid into two parts.]

Solution: The volume of the solid formed is given by:

$$\pi \int_{-1}^0 [(\sqrt{x+1})^2 - 0^2] dx + \pi \int_0^1 [(\sqrt{x+1})^2 - (\sqrt{2x})^2] dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Alternatively, the volume may be expressed as follows:

$$\pi \int_{-1}^1 [(\sqrt{x+1})^2 - 0^2] dx - \pi \int_0^1 [(\sqrt{2x})^2 - 0^2] dx = 2\pi - \pi = \pi.$$

3. (29) $V = \pi \int_a^b (f(x) - k)^2 dx$.

4. (39) A nose cone for a space reentry vehicle is designed so that a cross section, taken x feet from the tip and perpendicular to the axis of symmetry, is a circle of radius $\frac{1}{4}x^2$ feet. Find the volume of the nose cone given that its length is 20 feet.

Solution: Taking the tip at $x = 0$ and assuming that the height of the cone coincides with the interval $[0, 20]$ on the x -axis, we have $V = \pi \int_0^{20} \left(\frac{1}{4}x^2 \right)^2 dx = 40000\pi$ (cubic feet).

5. (40) A certain solid is 1 feet high, and a horizontal cross section taken x feet above the bottom of the solid is an annulus of inner radius x^2 feet and outer radius \sqrt{x} feet. Find the volume of the solid.

Solution: The volume of the solid is $V = \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2] dx = \frac{3\pi}{10}$ (cubic feet).

6. (28) Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = x^3$, $y = 1$, $x = 0$ is revolved about the line $y = 1$.

Solution: The volume of the solid $V = 2\pi \int_0^1 [(1-y)(y^{\frac{1}{3}} - 0)] dy = \frac{9\pi}{14}$.

7. (35) Consider a function $f(x)$ that is continuous and invertible on an interval $[a, b]$. Rotate about the y -axis the region bounded by $y = f(a)$, $x = b$, and $y = f(x)$. Then the values assigned to the volume of the resulting solid by applying either the disk method or the shell method coincide.

Equivalently: let D be the expression obtained using the disk (or washer) method

and S be the expression obtained using the shell method, i.e.

$$D = \pi \int_{f(a)}^{f(b)} [b^2 - (f^{-1}(y))^2] dy$$

and

$$S = 2\pi \int_a^b x[f(x) - f(a)] dx.$$

Then $D = S$.

8. (4) Find the exact arc length of the curve $x = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}}$ from $y = 0$ to $y = 1$.

Solution: The derivative of x with respect to y is $\frac{dx}{dy} = y\sqrt{y^2 + 2}$. The arc length is

$$S = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 \sqrt{1 + y^2(y^2 + 2)} dy = \frac{4}{3}.$$

9. (15) The region bounded by the graphs of $y = x^2$ and $y = x + 2$ is

$$\int_{-1}^2 (x + 2 - x^2) dx = \frac{9}{2}.$$

10. (17) The region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ is

$$\int_0^1 (\sqrt{x} - x^2) dx = \frac{1}{3}.$$