7 Dec 2011

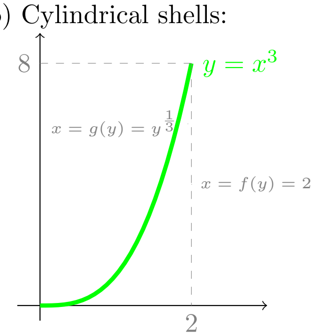
1. Let R be the region enclosed by the curves $y = x^2 + 4$, $y = x^3$, and the y-axis. Find and evaluate a definite integral that represents the volume of the solid generated by revolving R about the x-axis.

Solution: $V = \pi \displaystyle\int_0^2 [(x^2+4)^2 - (x^3)^2]dx = \pi\,\frac{4352}{105}$.

2. Consider the region enclosed by $y = x^3$, $y = 0$, and $x = 2$. Set up, but do not evaluate, an integral that represents the volume of the solid generated by revolving the region about the x-axis using (a) disks; (b) cylindrical shells.

Solution: (a) Disc method: $V = \pi \displaystyle\int_0^2 [(x^3)^2 - (0)^2]dx = \pi \int_0^2 x^6 dx \ (= \pi\frac{128}{7})$.

(b) Cylindrical shells:



$V = 2\pi \displaystyle\int_0^8 y[f(y) - g(y)]dy = 2\pi \int_0^8 y(2 - y^{\frac{1}{3}})dy \ (= \pi\frac{128}{7})$.

3. Let C be the curve $y = x^8$ between $x = 1$ and $x = 3$. In each part, set up, but do not evaluate, an integral that solves the problem.
(a) Find the arc length of C by integrating with respect to x.
(b) Find the arc length of C by integrating with respect to y.
Solution: (a) Consider an infinitesimal part of the curve ds. Then $ds^2 = dx^2 + dy^2$ (using the Pythagorean Theorem) and $\frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2}$, $ds = \sqrt{1 + (\frac{dy}{dx})^2}dx$. The length s of the part of the graph of $y = x^8$ between between $x = 1$ and $x = 3$ is

$$s = \int_1^3 \sqrt{1 + (\frac{dy}{dx})^2}dx = \int_1^3 \sqrt{1 + (8x^7)^2}dx.$$

(b) Consider $x = y^{\frac{1}{8}}$.

4. Find the centroid of the region bounded by $y^2 = 4x$ and $y^2 = 8(x - 2)$.
Solution: The area A of the bounded region is $A = \int_{-4}^4 [(2 + \frac{y^2}{8}) - \frac{y^2}{4}]dy = \frac{32}{3}$.
There are two moments, denoted by M_x and M_y. The moments measure the tendency of the region to rotate about the x and y-axis respectively. The moments are given by:

$$M_x = \int_{-4}^4 y[(2 + \frac{y^2}{8}) - \frac{y^2}{4}]dy = 0, \ M_y = \frac{1}{2}\int_{-4}^4 [(2 + \frac{y^2}{8})^2 - (\frac{y^2}{4})^2]dy = \frac{256}{15}.$$

The center of mass for this region is: $\bar{x} = \frac{M_y}{A} = \frac{8}{5} = 1.6$, $\bar{y} = \frac{M_x}{A} = 0$.