

30 Nov 2011

1. Find  $\int \sin x \cos^2 x dx$ .

$$\begin{aligned}\int \sin x \cos^2 x dx &= -\int u^2 du \text{ (letting } u = \cos x, du = -\sin x dx) \\ &= -\frac{u^3}{3} + C \\ &= -\frac{1}{3} \cos^3 x + C \text{ (where } C \text{ is the integration constant)}\end{aligned}$$

2. Find  $\int \sin^3 x \cos^2 x dx$ .

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx \\ &= \int (1 - \cos^2 x) \cos^2 x \sin x dx \text{ (using the identity } \sin^2 x + \cos^2 x = 1) \\ &= -\int (1 - u^2)u^2 du \text{ (letting } u = \cos x, du = -\sin x dx) \\ &= -\frac{u^3}{3} + \frac{u^5}{5} + C \\ &= -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C\end{aligned}$$

3. Find  $\int \cos(4\theta) \sqrt{2 - \sin(4\theta)} d\theta$ .

$$\begin{aligned}\int \cos(4\theta) \sqrt{2 - \sin(4\theta)} d\theta &= -\frac{1}{4} \int \sqrt{u} du \text{ (letting } u = 2 - \sin(4\theta), du = -4 \cos(4\theta) d\theta) \\ &= -\frac{1}{4} \cdot \frac{2}{3} \sqrt{u^3} + C \\ &= -\frac{1}{6} \sqrt{(2 - \sin(4\theta))^3} + C\end{aligned}$$

4. Find a horizontal line which divides the area between  $y = x^2$  and  $y = 9$  into two pieces of equal area.

Solution: The area surrounded by  $y = x^2$  ( $-3 \leq x \leq 3$ ) and  $y = 9$  is 36, so we need to find  $a \in (0, 9)$  such that

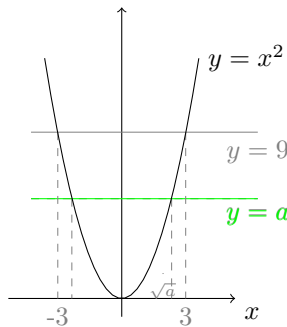
$$\int_{-\sqrt{a}}^{\sqrt{a}} a dx - \int_{-\sqrt{a}}^{\sqrt{a}} x^2 dx = 18.$$

We have:  $2a\sqrt{a} - \int_{-\sqrt{a}}^{\sqrt{a}} x^2 dx = 18$ ,

$$2a\sqrt{a} - \left(\frac{1}{3}x^3\right)\Big|_{-\sqrt{a}}^{\sqrt{a}} = 18,$$

$$2a\sqrt{a} - \left[\frac{1}{3}(\sqrt{a})^3 - \frac{1}{3}(-\sqrt{a})^3\right] = 18,$$

$$2a\sqrt{a} - \left[\frac{1}{3}a\sqrt{a} + \frac{1}{3}a\sqrt{a}\right] = 18,$$

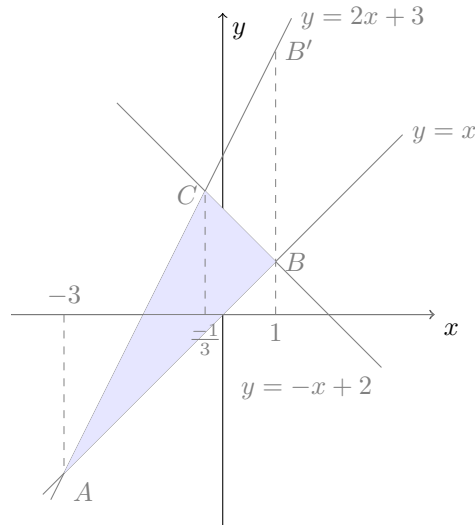


$$2a\sqrt{a} - \frac{2}{3}a\sqrt{a} = 18,$$

$\frac{4}{3}a\sqrt{a} = 18$ , so  $a = \left(\frac{27}{2}\right)^{\frac{2}{3}} = \frac{9}{\sqrt[3]{4}} \approx 5.7$ . The horizontal line which divides the area between  $y = x^2$  and  $y = 9$  into two pieces of equal area is  $y = \frac{9}{\sqrt[3]{4}}$ .

**5.** Find the area bounded by  $y = x$ ,  $y = 2x + 3$ ,  $y = -x + 2$ .

Solution: The area that we want to determine is the area of the triangle  $\triangle ABC$  (see the picture below).



Notice that  $\text{Area } \triangle ABC = \text{Area } \triangle ABB' - \text{Area } \triangle BB'C$ , so we have

$$\begin{aligned} \text{Area } \triangle ABC &= \int_{-3}^1 [(2x + 3) - x] dx - \int_{-\frac{1}{3}}^1 [(2x + 3) - (-x + 2)] dx \\ &= \int_{-3}^1 (x + 3) dx - \int_{-\frac{1}{3}}^1 (3x + 1) dx \\ &= 8 - \frac{8}{3} = \frac{16}{3} (\approx 5.3). \end{aligned}$$

Alternatively, you could notice that the area of the triangle  $\triangle ABC$  could also be written as:

$$\begin{aligned} \text{Area } \triangle ABC &= \int_{-3}^{-\frac{1}{3}} [(2x + 3) - x] dx + \int_{-\frac{1}{3}}^1 [(-x + 2) - x] dx \\ &= \int_{-3}^{-\frac{1}{3}} (x + 3) dx + \int_{-\frac{1}{3}}^1 (-2x + 2) dx \\ &= \frac{32}{9} + \frac{16}{9} = \frac{16}{3}. \end{aligned}$$

**21.** Water is run at a constant rate of  $1 \text{ ft}^3/\text{min}$  to fill a cylindrical tank of radius 3 ft and height 5 ft. Assuming that the tank is initially empty, make a conjecture about the average weight of the water in the tank over the time period required to fill it, and then check your conjecture by integrating. [Take the weight density of water is  $62.4 \text{ lb}/\text{ft}^3$ .]

Solution:

- (a) The volume of the tank is  $\pi r^2 h = \pi 3^2 \cdot 5 = 45\pi \approx 141 \text{ ft}^3$ .
- (b) Since water is run at a constant rate of  $1 \text{ ft}^3/\text{min}$ , (the approximate) time to fill the tank completely is 141 minutes (= 2h + 21min).
- (c) The weight of water can be obtained as a product of water-volume and density, so when the tank is full the weight of water is

$$W = 45\pi \cdot 62.4 \approx 8822 \text{ lb.}$$

- (d) Conjecture: the average weight of the water in the tank over the time period required to fill it is

$$W_{\text{avg}} = \frac{45\pi \cdot 62.4}{2} \approx 4411 \text{ lb.}$$

Check the conjecture: After time  $t$  ( $0 \leq t \leq 141$  minutes), the water-volume in the tank is  $V(t) = t \cdot 1 = t \text{ ft}^3$ , and the water-weight is  $W(t) = V(t) \cdot 62.4 = 62.4t$ . The average value of the function  $W(t)$  from  $t = 0$  to  $t = 45\pi (\approx 141)$  is given by

$$\begin{aligned} W_{\text{avg}} &= \frac{1}{45\pi - 0} \int_{t=0}^{t=45\pi} W(t) dt \\ &= \frac{1}{45\pi} \int_{t=0}^{t=45\pi} 62.4t dt \\ &= \frac{62.4}{45\pi} \int_{t=0}^{t=45\pi} t dt \\ &= \frac{62.4}{45\pi} \left( \frac{t^2}{2} \right) \Big|_{t=0}^{t=45\pi} \\ &= \frac{62.4 (45\pi)^2}{45\pi \cdot 2} \\ &= \frac{62.4 \cdot 45\pi}{2} \approx 4411 \text{ lb.} \end{aligned}$$

**23.** A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data, she estimates that between 4:30 p.m. and 5:30 p.m. the rate  $R(t)$  at which cars enter the highway is given by the formula  $R(t) = 100(1 - 0.0001t^2)$  cars per minute, where  $t$  is the time (in minutes) since 4:30 p.m. Find the average rate, in cars per minute, at which cars enter the highway during the first half-hour of rush hour.

Solution: The average rate, in cars per minute, at which cars enter the highway during the first half-hour of rush hour is the average value of the function  $R(t)$  from  $t = 0$  to  $t = 30$  minutes, so it is given by:

$$\frac{1}{30 - 0} \int_0^{30} R(t) dt = \frac{1}{30} \int_0^{30} 100(1 - 0.0001t^2) dt = \frac{2910}{30} = 97.$$

On average, cars enter the highway during the first half-hour of rush hour at a rate of 97 cars per minute.

**24.** Suppose that the value of a yacht in dollars after  $t$  years of use is

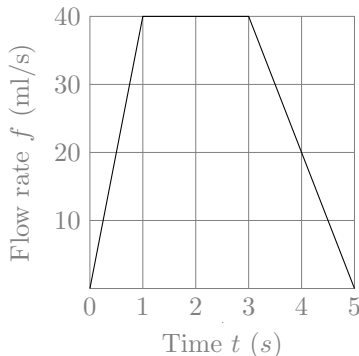
$$V(t) = 275,000\sqrt{\frac{20}{t+20}}.$$

What is the average value of the yacht over its first 10 years of use?

Solution: The average value of the yacht over its first 10 years of use is

$$\frac{1}{10} \int_0^{10} V(t) dt = \frac{275,000}{10} \int_0^{10} \sqrt{\frac{20}{t+20}} = \frac{275,000}{10} \cdot 20(\sqrt{6}-2) = 247,219 \text{ [dollars]}.$$

**25.** A large juice glass containing 60 ml of orange juice is replenished by a server. The accompanying figure shows the rate at which orange juice is poured into the glass in milliliters per second (ml/s). Show that the average rate of change of the volume of juice in the glass during these 5 s is equal to the average value of the rate of flow of juice into the glass.



Solution:

$$f_{\text{avg}} = \frac{1}{5} \left[ \int_0^1 40t dt + \int_1^3 40 dt + \int_3^5 (-20t + 100) dt \right] = \frac{1}{5} [20 + 80 + 40] = 28.$$

The rate of change of the volume of juice in the glass is given by

$$V(t) = \begin{cases} 20t^2 + 60 & \text{if } 0 \leq t \leq 1 \\ 40t + 40 & \text{if } 1 \leq t \leq 3 \\ -10t^2 + 100t - 50 & \text{if } 3 \leq t \leq 5 \end{cases}$$

so the average  $V$  is  $\frac{V(5)-V(0)}{5} = \frac{200-60}{5} = 28$ .