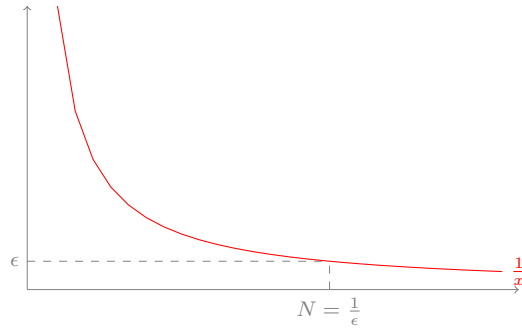


Exercise 13. Prove $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$.

Need to prove:

$$\forall \epsilon > 0 \quad \exists N > 0 \quad \forall x \in \mathbb{R} \quad \left(x > N \implies \left| \frac{1}{x} - 0 \right| < \epsilon \right).$$



Fix $\epsilon > 0$ and let $N = \frac{1}{\epsilon}$. Then for all $x > N$ we have $\left| \frac{1}{x} \right| = \frac{1}{x} < \frac{1}{N} = \epsilon$.

Exercise 14. For $x \neq -\frac{1}{3}$ we have:

$$\begin{aligned} \left| \frac{28}{3x+1} - 4 \right| &= \left| \frac{28}{3x+1} - \frac{4(3x+1)}{3x+1} \right| \\ &= \left| \frac{24 - 12x}{3x+1} \right| \\ &= \left| \frac{12x - 24}{3x+1} \right| \\ &= \left| \frac{12}{3x+1} \right| |x - 2|. \end{aligned}$$

If $|x - 2| < 4$, i.e. $-2 < x < 6$, then the function $\frac{12}{3x+1}$ ($x \neq -\frac{1}{3}$) is not bounded since $\lim_{x \rightarrow (-\frac{1}{3})^-} \frac{12}{3x+1} = -\infty$, $\lim_{x \rightarrow (-\frac{1}{3})^+} \frac{12}{3x+1} = +\infty$. However, if $|x - 2| < 1$, i.e. $1 < x < 3$, the function $\frac{12}{3x+1}$ ($x \neq -\frac{1}{3}$) is bounded. More specifically, if $1 < x < 3$ then $\frac{12}{10} < \frac{12}{3x+1} < 3$.

To prove $\lim_{x \rightarrow 2} \frac{28}{3x+1} = 4$, we need to prove the following:

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in \mathbb{R} \quad \left(0 < |x - 2| < \delta \implies \left| \frac{28}{3x+1} - 4 \right| < \epsilon \right).$$

1. Fix $\epsilon > 0$.

2. To find δ note: $0 < |x - 2| < \delta$ if and only if $x \neq 2$ and $2 - \delta < x < 2 + \delta$. Consider only those δ 's that are at most 1, i.e. $\delta \in (0, 1]$. Then $1 < x < 3$ (and $x \neq 2$). Now, using the arguments given above we obtain

$$\left| \frac{28}{3x+1} - 4 \right| = \left| \frac{12}{3x+1} \right| |x - 2| < 3 \cdot \delta$$

since the function $\frac{12}{3x+1}$ ($x \in (1, 3)$) is bounded by 3. Take $\delta = \min\{1, \frac{\epsilon}{3}\}$.

3. Then for $x \in \mathbb{R}$ with $0 < |x - 2| < \delta$ we have $\left| \frac{28}{3x+1} - 4 \right| = \left| \frac{12}{3x+1} \right| |x - 2| < 3 \cdot \delta \leq \epsilon$.