1. Let

$$f(x) = \begin{cases} \frac{x}{\sqrt{x^2+a^4-a^2}} & \text{if } x > 0, \\ b & \text{if } x = 0, \\ \frac{\sin ax}{x} & \text{if } x < 0. \end{cases}$$

Find the values of a and b that make f continuous at 0.

Solution: Since the expression $x^2 + a^4 - a^2$ is in a denominator and "under" a square root, we must have

$$x^2 + a^4 - a^2 > 0 \qquad \text{for all } x > 0,$$

or equivalently

$$x^2 > a^2 - a^4 \qquad \text{for all } x > 0.$$

Hence we must have: $0 \geq a^2 - a^4$, $a^4 - a^2 \geq 0$; that is $a^4 - a^2 > 0$ or $a^4 - a^2 = 0$.

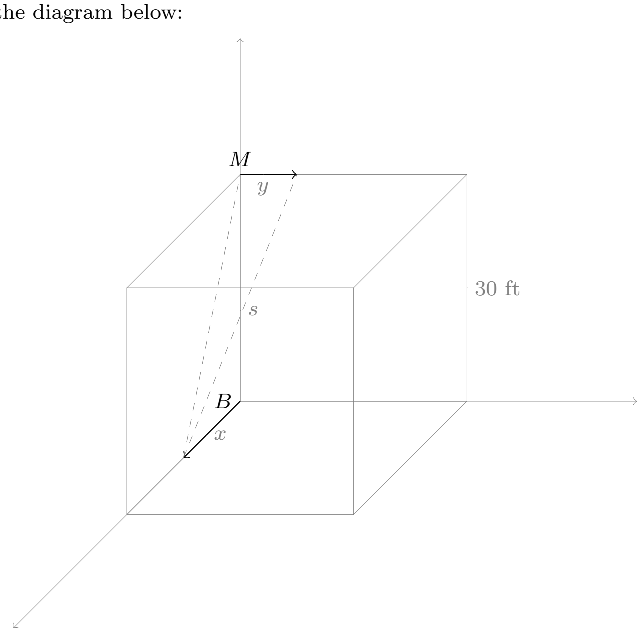
Now the right-hand limit of f as x approaches 0 is

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{x}{\sqrt{x^2+a^4-a^2}} \\ &= \lim_{x\to 0^+} \frac{x}{\sqrt{x^2(1+\frac{a^4-a^2}{x^2})}} \\ &= \lim_{x\to 0^+} \frac{1}{\sqrt{1+\frac{a^4-a^2}{x^2}}} = \begin{cases} 0 & \text{if } a^4 - a^2 > 0 \\ 1 & \text{if } a^4 - a^2 = 0, \end{cases} \end{aligned}$$

and the left-hand limit of f as x approaches 0 is $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin ax}{x} = a$. But f is continuous at 0 if and only if the right- and left-hand limits exist and both equal $f(0) = b$. So to make f continuous at 0 take $a = b = 1$.

2. A bridge is 30 ft. above a canal. A boat B going 10 ft/sec passes under the center of the bridge at the same instant that a man M walking 5 ft/sec reaches that point. How rapidly are they separating 3 sec. later? How fast is the distance between the man and the boat separating 3 seconds later?

Solution: In the diagram below:



Let M be the position of the man, and let B be the position of the boat. Let $x = x(t)$ be the distance traveled by B in time t, and let $y = y(t)$ be the distance traveled by M in time t. Let $s = s(t)$ be the distance separating B and M in time t. Then

$$s^2 = x^2 + y^2 + (30)^2 = x^2 + y^2 + 900. \qquad \text{(Here } s, x, y \text{ are functions of time.)}$$

Differentiate both sides of the equation above with respect to time, t, to get

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Dividing by 2 gives $s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$, and dividing by s gives

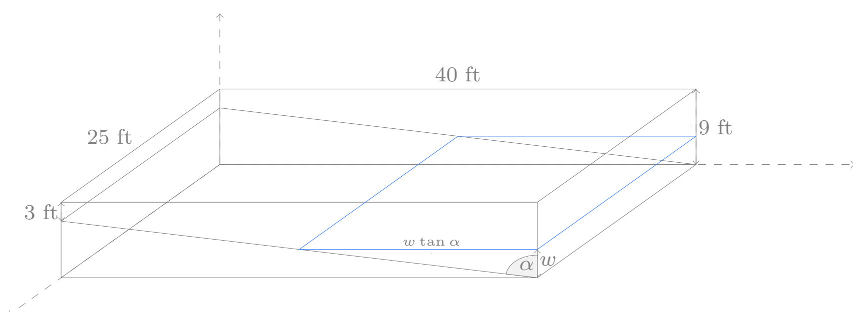
$$\frac{ds}{dt} = \frac{1}{s}(x\frac{dx}{dt} + y\frac{dy}{dt}).$$

After 3 seconds: $x = 30$, $y = 15$, $s = \sqrt{(30)^2 + (15)^2 + 900} = 45$ ft, and

$$\frac{ds}{dt} = \frac{1}{45}(30 \cdot 10 + 15 \cdot 5) = \frac{25}{3},$$

so the distance between the man and the boat is increasing at a rate of $\frac{25}{3} \approx 8.3$ ft/sec.

3. A swimming pool is 25 ft. wide, 40 ft. long, 3 ft. deep at the shallow end, and 9 ft. deep at the deep end, the bottom being an inclined plane. If water is pumped into the pool at the rate of 10 ft^3/min, how fast is the water level rising when it is 4 ft. deep at the deep end? *Solution:* In the diagram:



Let $w = w(t)$ be the water level at time t $(0 \le w \le 9)$. Then $\tan\alpha = \frac{\cos(\frac{\pi}{2}-\alpha)}{\sin(\frac{\pi}{2}-\alpha)} = \frac{40}{6} = \frac{20}{3}$, and the volume $V = V(t)$ of the water in the swimming pool (in time t when it is partly filled) is

$$\begin{aligned} V &= \begin{cases} \text{the area of the triangle "}\alpha\text{" times the width} & \text{if } 0 \le w \le 6 \\ \text{the area of a trapezoid times the width} & \text{if } 6 < w \le 9 \end{cases} \\ &= \begin{cases} \frac{1}{2}w \cdot w\tan\alpha \cdot 25 & \text{if } 0 \le w \le 6 \\ \text{the area of a trapezoid times the width} & \text{if } 6 < w \le 9 \end{cases} \\ &= \begin{cases} \frac{250}{3}w^2 & \text{if } 0 \le w \le 6 \\ \text{the area of a trapezoid times the width} & \text{if } 6 < w \le 9. \end{cases} \end{aligned}$$

Since we are only interested in those values w which are close to 4 ("... how fast is the water level rising when it is 4 ft. deep at the deep end?"), we should confine our attention to the first formula for V and consider

$$V = \frac{250}{3}w^2 \qquad \text{(where } 0 \le w \le 6\text{)}.$$

Differentiating $V = \frac{250}{3}w^2$ with respect to time t gives $\frac{dV}{dt} = \frac{500}{3}w\frac{dw}{dt}$. Multiplying this by $\frac{3}{500}$ gives $\frac{3}{500}\frac{dV}{dt} = w\frac{dw}{dt}$, and dividing by w gives $\frac{dw}{dt} = \frac{3}{500w}\frac{dV}{dt}$. Since $\frac{dV}{dt} = 10$, at $w = 4$ we have $\frac{dw}{dt} = \frac{3}{200}$. Thus, if water is pumped into the pool at the rate of 10 ft^3/min, the water level rising at a rate of $\frac{3}{200}$ ft/min when it is 4 ft. deep at the deep end.

4. Let

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous at 0.

Solution: No. Consider the sequences

$$a_n = \frac{2}{\pi + 4n\pi}, \qquad b_n = \frac{2}{3\pi + 4n\pi} \qquad\qquad (n = 0, 1, 2, \dots).$$

Then $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$, but $\lim_{n\to\infty} f(a_n) = 1 \neq -1 = \lim_{n\to\infty} f(b_n)$. Thus f is not continuous at 0.