Part III. Calculus of variations

Lecture notes for MA342H

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Introduction

- There are several applications that involve expressions of the form

$$J(y) = \int_a^b L(x, y(x), y'(x))\, dx.$$

  For instance, $J(y)$ could represent area, length, energy and so on.
- We are interested in finding the minimum/maximum values of $J(y)$ over all functions $y(x)$ that satisfy some given constraints.
- A function $y(x)$ for which the minimum/maximum value is attained is called an extremal of $J(y)$. We shall assume that both $y(x)$ and the Lagrangian function L are twice continuously differentiable.
- One may impose various constraints on $y(x)$, but the most common one amounts to specifying its values at the endpoints $x = a, b$. This requires the graph of $y(x)$ to pass through two given points.

Directional derivative

- Consider a functional of the form

$$J(y) = \int_a^b L(x, y(x), y'(x))\, dx.$$

- Its directional derivative in the direction of a function φ is defined as

$$J'(y)\varphi = \lim_{\varepsilon \to 0} \frac{J(y + \varepsilon\varphi) - J(y)}{\varepsilon}.$$

  This is also known as the first variation of $J(y)$. Explicitly, one has

$$J'(y)\varphi = \int_a^b \left(L_y \varphi + L_{y'} \varphi' \right)\, dx.$$

- The function φ is called a test function, if it is twice continuously differentiable and also vanishing at the endpoints $x = a, b$.

Directional derivative: Proof

- The directional derivative $J'(y)\varphi$ is defined in terms of

$$J(y+\varepsilon\varphi) - J(y) = \int_a^b \Big(L(x, y+\varepsilon\varphi, y'+\varepsilon\varphi') - L(x,y,y') \Big)\, dx.$$

- Denote the integrand by $F(\varepsilon)$. Then a Taylor series expansion gives

$$F(\varepsilon) = L_y(x,y,y') \cdot \varepsilon\varphi + L_{y'}(x,y,y') \cdot \varepsilon\varphi' + \ldots,$$

  where the dots indicate terms which contain higher powers of ε.

- Once we now combine the last two equations, we find that

$$\begin{aligned} J'(y)\varphi &= \lim_{\varepsilon\to 0} \frac{J(y+\varepsilon\varphi) - J(y)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \int_a^b \frac{F(\varepsilon)}{\varepsilon}\, dx = \int_a^b \left(L_y\varphi + L_{y'}\varphi' \right)\, dx. \end{aligned}$$

Euler-Lagrange equation

Theorem 1. Euler-Lagrange equation

Suppose that $y(x)$ is an extremal of

$$J(y) = \int_a^b L(x, y(x), y'(x))\, dx.$$

Then $y(x)$ must satisfy the Euler-Lagrange equation

$$\frac{d}{dx} L_{y'} = L_y.$$

- A solution of the Euler-Lagrange equation is also known as a critical point of $J(y)$. Critical points of $J(y)$ are not necessarily extremals.
- When the Lagrangian L depends on higher-order derivatives of y, the Euler-Lagrange equation involves higher-order derivatives of L.

Euler-Lagrange equation: Proof

- Since $y(x)$ is an extremal of $J(y)$, it must satisfy

$$J'(y)\varphi = 0 \quad \Longrightarrow \quad \int_a^b \left(L_y\varphi + L_{y'}\varphi'\right)\,dx = 0$$

for every test function φ. Integrating by parts, we now get

$$\int_a^b \left(L_y - \frac{d}{dx}L_{y'}\right)\varphi\,dx + \Big[L_{y'}\varphi\Big]_a^b = 0.$$

- Since the test function φ vanishes at the endpoints, this gives

$$\int_a^b \left(L_y - \frac{d}{dx}L_{y'}\right)\varphi\,dx = 0$$

for every test function φ. According to the fundamental lemma of variational calculus, we must thus have $L_y = \frac{d}{dx}L_{y'}$, as needed.

Fundamental lemma of variational calculus

- Suppose that $H(x)$ is continuously differentiable with

$$\int_a^b H(x)\varphi(x)\,dx = 0$$

for every test function φ. Then $H(x)$ must be identically zero.

- To prove this, consider an arbitrary subinterval $[x_1, x_2]$ and let

$$\varphi(x) = \left\{ \begin{array}{cc} (x-x_1)^3(x_2-x)^3 & \text{if } x_1 \le x \le x_2 \\ 0 & \text{otherwise} \end{array} \right\}.$$

Then φ is a test function which is positive on (x_1, x_2) and we have

$$0 = \int_a^b H(x)\varphi(x)\,dx = \int_{x_1}^{x_2} H(x)\varphi(x)\,dx.$$

If $H(x)$ is positive/negative at a point, it must have the same sign on some interval $[x_1, x_2]$ and this is not the case by the last equation.

Beltrami identity

- Suppose that $y(x)$ is a critical point of the functional

$$J(y) = \int_a^b L(y(x), y'(x))\, dx$$

whose Lagrangian function L does not depend on x directly. Then

$$y' L_{y'} - L = \text{constant}.$$

- To prove this, one uses the Euler-Lagrange equation to find that

$$\frac{d}{dx}\left(y' L_{y'}\right) = y'' L_{y'} + y' \frac{d}{dx} L_{y'} = y'' L_{y'} + y' L_y.$$

Since $L(y, y')$ does not depend on x directly, the chain rule gives

$$\frac{d}{dx}\left(y' L_{y'}\right) = y'' L_{y'} + y' L_y = \frac{d}{dx} L.$$

Example 1. Shortest path

- Consider a function $y(x)$ whose graph passes through two given points. We wish to minimise the length of its graph

$$J(y) = \int_a^b \sqrt{1 + y'(x)^2}\, dx.$$

- Since $L = \sqrt{1 + y'(x)^2}$, the Euler-Lagrange equation gives

$$\frac{d}{dx} L_{y'} = L_y = 0 \quad \Longrightarrow \quad L_{y'} = \frac{y'(x)}{\sqrt{1 + y'(x)^2}} = c_1.$$

  We square both sides and then simplify to conclude that

$$y'(x)^2 = c_1^2 + c_1^2 y'(x)^2 \quad \Longrightarrow \quad y'(x) = c_2.$$

- This shows that $y(x)$ must be a linear function. In other words, the shortest path between any two points is given by a line.

Example 2. Minimal surface, page 1

- Consider a function $y(x)$ whose graph passes through two given points. We wish to minimise the surface area of the solid which is obtained by rotating the graph of $y(x)$ around the x-axis, namely

$$J(y) = \int_a^b 2\pi y(x)\sqrt{1 + y'(x)^2}\, dx.$$

- Since the Lagrangian does not depend on x directly, one has

$$c = y' L_{y'} - L = \frac{2\pi y(x) y'(x)^2}{\sqrt{1 + y'(x)^2}} - 2\pi y(x)\sqrt{1 + y'(x)^2}$$

by the Beltrami identity. It now easily follows that

$$\frac{c}{2\pi} = -\frac{y(x)}{\sqrt{1 + y'(x)^2}} \quad\Longrightarrow\quad 1 + y'(x)^2 = \frac{y(x)^2}{a^2}.$$

Example 2. Minimal surface, page 2

- The last equation is actually a separable equation which gives

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{a^2} - 1} \quad \Longrightarrow \quad \int \frac{dy}{\sqrt{y^2 - a^2}} = \int \frac{dx}{a}.$$

- Letting $y = a\cosh t$, we now get $y^2 - a^2 = a^2 \sinh^2 t$, hence also

$$\int \frac{dx}{a} = \int \frac{a\sinh t\, dt}{a\sinh t} = \int dt.$$

- Since $y = a\cosh t$ by above, we may finally conclude that

$$\frac{x - x_0}{a} = t \quad \Longrightarrow \quad y = a\cosh t = a\cosh\left(\frac{x - x_0}{a}\right).$$

  This is an equation that describes the shape of a hanging chain.

Example 3. Brachistochrone, page 1

- Consider a particle that slides along the graph of a function $y(x)$ from one point to another under the influence of gravity. We assume that there is no friction and wish to minimise the overall travel time

$$J(y) = \int_a^b \frac{\sqrt{1 + y'(x)^2}\, dx}{v(y(x))}.$$

- To find the speed v, we note that conservation of energy gives

$$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 + mgy_0.$$

- Assuming that $v_0 = 0$ for simplicity, the overall travel time is then

$$J(y) = \int_a^b \frac{\sqrt{1 + y'(x)^2}\, dx}{\sqrt{2g(y_0 - y)}}.$$

Example 3. Brachistochrone, page 2

- Since the Lagrangian does not depend on x directly, one has

$$c = y' L_{y'} - L = \frac{y'(x)^2}{\sqrt{2g(y_0 - y)}\sqrt{1 + y'(x)^2}} - \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2g(y_0 - y)}}$$

by the Beltrami identity. We clear denominators to get

$$c\sqrt{2g(y_0 - y)}\sqrt{1 + y'(x)^2} = -1$$

and then we square both sides to find that

$$1 + y'(x)^2 = \frac{a^2}{y_0 - y(x)} \quad \Longrightarrow \quad y'(x)^2 = \frac{a^2 - y_0 + y(x)}{y_0 - y(x)}.$$

- This is a separable equation that can also be written as

$$\int \frac{\sqrt{y_0 - y}\, dy}{\sqrt{a^2 - y_0 + y}} = \int dx.$$

Example 3. Brachistochrone, page 3

- Letting $y_0 - y = a^2 \sin^2\theta$ gives $a^2 - y_0 + y = a^2\cos^2\theta$, hence

$$
\begin{aligned}
-\int dx &= -\int \frac{\sqrt{y_0 - y}\, dy}{\sqrt{a^2 - y_0 + y}} \\
&= \int \frac{a\sin\theta \cdot 2a^2 \sin\theta\cos\theta\, d\theta}{a\cos\theta} = a^2 \int (1 - \cos(2\theta))\, d\theta.
\end{aligned}
$$

- Once we now simplify the integrals, we may finally conclude that

$$
\begin{aligned}
x_0 - x &= a^2\left(\theta - \frac{1}{2}\sin(2\theta)\right) = \frac{a^2}{2}(\varphi - \sin\varphi), \\
y_0 - y &= a^2\sin^2\theta = \frac{a^2}{2}(1 - \cos(2\theta)) = \frac{a^2}{2}(1 - \cos\varphi),
\end{aligned}
$$

where $\varphi = 2\theta$. These are the parametric equations of a cycloid.

Case 1. Fixed boundary conditions

- Suppose that we wish to find the extremals of

$$J(y) = \int_a^b L(x, y(x), y'(x))\,dx$$

  subject to given boundary conditions $y(a) = y_0$ and $y(b) = y_1$.

- This is the standard variational problem which implies that

$$\int_a^b \left(L_y - \frac{d}{dx} L_{y'} \right) \varphi\,dx + \Big[L_{y'} \varphi \Big]_a^b = 0$$

  for every test function φ. Since the boundary terms vanish, one can find the possible extremals by solving the Euler-Lagrange equation

$$\frac{d}{dx} L_{y'} = L_y.$$

- Thus, we get a second-order ODE subject to two boundary conditions.

Case 1. Example

- As a simple example, consider the functional

$$J(y) = \int_0^1 y'(x)^2 \, dx$$

subject to the boundary conditions $y(0) = 1$ and $y(1) = 3$.

- In this case, the Euler-Lagrange equation gives

$$\frac{d}{dx} L_{y'} = L_y \quad \Longrightarrow \quad 2y''(x) = 0 \quad \Longrightarrow \quad y(x) = ax + b.$$

- To ensure that the boundary conditions hold, we must have

$$1 = y(0) = b, \qquad 3 = y(1) = a + b.$$

Thus, the only possible extremal is given by $y(x) = 2x + 1$.

Case 2. Variable boundary conditions

- Suppose that we wish to find the extremals of

$$J(y) = \int_a^b L(x, y(x), y'(x))\, dx$$

when no boundary conditions are specified for $y(a)$ and $y(b)$.

- Using the same approach as before, one finds that

$$\int_a^b \left(L_y - \frac{d}{dx} L_{y'} \right) \varphi \, dx + \Big[L_{y'} \varphi \Big]_a^b = 0$$

for every function φ. This obviously includes all test functions, so the Euler-Lagrange equation remains valid, but we must also have

$$L_{y'} = 0 \text{ when } x = a, b.$$

- These conditions are known as the natural boundary conditions.

- As a typical example, consider the functional

$$J(y) = \int_0^1 \Big(y'(x)^2 + y(x)y'(x) - 4y(x) \Big)\, dx.$$

- Using the Euler-Lagrange equation, one finds that

$$\begin{aligned} \frac{d}{dx} L_{y'} = L_y \quad &\Longrightarrow \quad 2y'' + y' = y' - 4 \\ &\Longrightarrow \quad y(x) = -x^2 + ax + b. \end{aligned}$$

- In view of the natural boundary conditions, we must also have

$$0 = L_{y'} = 2y' + y = 2(a - 2x) - x^2 + ax + b$$

when $x = 0, 1$. This gives $2a + b = 0 = 3a + b - 5$, so we easily find that $a = 5$ and $b = -10$. In other words, $y(x) = -x^2 + 5x - 10$.

Case 3. Several unknown functions

- Suppose that we wish to find the extremals of

$$J(y,z) = \int_a^b L(x, y(x), y'(x), z(x), z'(x))\, dx$$

subject to given boundary conditions, say

$$y(a) = y_0, \qquad y(b) = y_1, \qquad z(a) = z_0, \qquad z(b) = z_1.$$

- Viewing $J(y,z)$ as a function of one variable, we must then have

$$\frac{d}{dx} L_{y'} = L_y, \qquad \frac{d}{dx} L_{z'} = L_z.$$

- In particular, one can find the extremals by solving a system of two second-order equations subject to four boundary conditions.

Case 3. Example

- As a typical example, consider the functional

$$J(y,z) = \int_0^1 \left(y'(x)z'(x) + y(x)^2 \right) dx$$

subject to the conditions $y(0) = z(0) = 0$ and $y(1) = z(1) = 1$.

- The corresponding Euler-Lagrange equations are then

$$\begin{aligned} \frac{d}{dx} L_{y'} = L_y \quad &\Longrightarrow \quad z'' = 2y, \\ \frac{d}{dx} L_{z'} = L_z \quad &\Longrightarrow \quad y'' = 0. \end{aligned}$$

- Solving the latter gives $y = ax + b$, hence also $y = x$. Solving the former, we now get $z'' = 2x$, so $z = \frac{1}{3}x^3 + cx + d = \frac{1}{3}(x^3 + 2x)$.

Case 4. Isoperimetric constraints

- Suppose that we wish to find the extremals of

$$J(y) = \int_a^b L(x, y(x), y'(x))\, dx$$

subject to the boundary and integral constraints

$$y(a) = y_0, \qquad y(b) = y_1, \qquad \int_a^b M(x, y(x), y'(x))\, dx = c.$$

- Let us denote by $I(y)$ the integral that appears in the last equation. Then the extremals of $J(y)$ are either critical points of $I(y)$ or else critical points of $J(y) - \lambda I(y)$ for some Lagrange multiplier $\lambda \in \mathbb{R}$.
- In particular, one has to solve the Euler-Lagrange equation for two different Lagrangians, namely M and also $L - \lambda M$.

- We determine the possible extremals of

$$J(y) = \int_0^\pi y'(x)^2 \, dx$$

subject to the boundary and integral constraints

$$y(0) = y(\pi) = 0, \qquad \int_0^\pi y(x) \sin x \, dx = 1.$$

- Let us denote by $I(y)$ the integral that appears in the last equation. Its critical points must satisfy the Euler-Lagrange equation

$$L = y(x) \sin x \quad \Longrightarrow \quad \frac{d}{dx} L_{y'} = L_y \quad \Longrightarrow \quad 0 = \sin x$$

and this means that $I(y)$ has no critical points at all.

Case 4. Example, page 2

- To find the critical points of $J(y) - \lambda I(y)$, we note that

$$L = y'(x)^2 - \lambda y(x)\sin x \quad \Longrightarrow \quad \frac{d}{dx} L_{y'} = L_y$$
$$\Longrightarrow \quad 2y'' = -\lambda \sin x.$$

- Integrating this equation twice, we conclude that

$$y'(x) = \frac{\lambda}{2}\cos x + a \quad \Longrightarrow \quad y(x) = \frac{\lambda}{2}\sin x + ax + b.$$

Since $y(0) = y(\pi) = 0$, it is easy to check that $a = b = 0$. Thus, it remains to determine λ. Using the integral constraint, we get

$$1 = \int_0^\pi y(x)\sin x\,dx = \frac{\lambda}{2}\int_0^\pi \sin^2 x\,dx = \frac{\lambda\pi}{4}.$$

- This gives $\lambda = \frac{4}{\pi}$, so the only possible extremal is $y(x) = \frac{2}{\pi}\sin x$.

Lagrange multipliers

Theorem 2. Lagrange multipliers

Suppose that $y(x)$ is an extremal of

$$J(y) = \int_a^b L(x, y(x), y'(x))\, dx$$

subject to the integral constraint $I(y) = c$, where

$$I(y) = \int_a^b M(x, y(x), y'(x))\, dx.$$

Then the extremal $y(x)$ must be either a critical point of $I(y)$ or else a critical point of $J(y) - \lambda I(y)$ for some Lagrange multiplier $\lambda \in \mathbb{R}$.

- This theorem is closely related to the corresponding theorem for the extrema of a function $f(x, y)$ subject to a constraint $g(x, y) = c$.

Lagrange multipliers: Proof, page 1

- Let φ, ψ be some given test functions and define $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\varepsilon, \delta) = J(y + \varepsilon\varphi + \delta\psi), \qquad g(\varepsilon, \delta) = I(y + \varepsilon\varphi + \delta\psi).$$

- We note that $g(0,0) = I(y) = c$ by assumption, while

$$g_\varepsilon(0,0) = \lim_{\varepsilon \to 0} \frac{g(\varepsilon, 0) - g(0,0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{I(y + \varepsilon\varphi) - I(y)}{\varepsilon} = I'(y)\varphi.$$

- Suppose $y(x)$ is not a critical point of $I(y)$. We can then find a test function φ such that $I'(y)\varphi \neq 0$. According to the implicit function theorem, we can thus find a function $\varepsilon = \varepsilon(\delta)$ with $\varepsilon(0) = 0$ and

$$c = g(\varepsilon, \delta) = I(y + \varepsilon\varphi + \delta\psi)$$

in a neighbourhood of $\delta = 0$, namely for small enough δ.

Lagrange multipliers: Proof, page 2

- Since $y(x)$ is an extremal of $J(y)$ subject to the given constraint,

$$f(\varepsilon, \delta) = J(y + \varepsilon\varphi + \delta\psi)$$

attains an extremum at $(0, 0)$ subject to the constraint $g(\varepsilon, \delta) = c$.

- Using a standard calculus result, we conclude that either $\nabla g(0, 0) = 0$ or else $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ for some $\lambda \in \mathbb{R}$. One may easily exclude the former case since $g_\varepsilon(0, 0) = I'(y)\varphi \neq 0$ by above. We thus have

$$\begin{aligned} \nabla f(0,0) = \lambda \nabla g(0,0) \quad &\implies \quad f_\delta(0,0) = \lambda g_\delta(0,0) \\ &\implies \quad J'(y)\psi = \lambda I'(y)\psi \end{aligned}$$

for all test functions ψ, so $y(x)$ is a critical point of $J(y) - \lambda I(y)$.

Second variation

- The first variation of $J(y)$ is defined as the limit

$$J'(y)\varphi = \lim_{\varepsilon \to 0} \frac{J(y+\varepsilon\varphi) - J(y)}{\varepsilon}$$

and one can use a Taylor series expansion to derive the formula

$$J'(y)\varphi = \int_a^b \left(L_y \varphi + L_{y'} \varphi' \right) \, dx.$$

- The second variation of $J(y)$ is defined as the limit

$$J''(y)\varphi = \lim_{\varepsilon \to 0} \frac{J(y+\varepsilon\varphi) - J(y) - \varepsilon J'(y)\varphi}{\frac{1}{2}\varepsilon^2}$$

and one can use a Taylor series expansion to derive the formula

$$J''(y)\varphi = \int_a^b \Big(L_{yy}\varphi^2 + 2L_{yy'}\varphi\varphi' + L_{y'y'}(\varphi')^2 \Big) \, dx.$$

Second variation: Sketch of proof

- According to the definition of $J(y)$, one has

$$J(y+\varepsilon\varphi) - J(y) = \int_a^b \Big(L(x, y+\varepsilon\varphi, y'+\varepsilon\varphi') - L(x,y,y') \Big)\, dx.$$

- Denote the integrand by $F(\varepsilon)$. Then a Taylor series expansion gives

$$\begin{aligned} F(\varepsilon) = \varepsilon\Big(& L_y\varphi + L_{y'}\varphi'\Big) \\ &+ \frac{\varepsilon^2}{2}\Big(L_{yy}\varphi^2 + 2L_{yy'}\varphi\varphi' + L_{y'y'}(\varphi')^2 \Big) + \ldots, \end{aligned}$$

  where the dots indicate terms which contain higher powers of ε.

- Since the linear terms correspond to the first variation $J'(y)\varphi$, it easily follows that the quadratic terms correspond to $J''(y)\varphi$.

Theorem 3. Necessary condition

If the functional $J(y)$ attains a local minimum at the function $y(x)$, then we must actually have $J''(y)\varphi \geq 0$ for all functions φ.

- This condition resembles the second derivative test from advanced calculus. It is closely related to the expansion

$$J(y + \varepsilon\varphi) = J(y) + \varepsilon J'(y)\varphi + \frac{\varepsilon^2}{2} J''(y)\varphi + \ldots$$

  and the fact that $J'(y)\varphi = 0$ for all critical points of $J(y)$.
- It may happen that $J''(y)\varphi > 0$ for all functions φ, even though $J(y)$ has no local minimum. In particular, the given condition is necessary but not sufficient for the existence of a local minimum.

Necessary condition: Sketch of proof

- Using the definition of $J''(y)\varphi$, one can write

$$J(y+\varepsilon\varphi) = J(y) + \varepsilon J'(y)\varphi + \frac{\varepsilon^2}{2} J''(y)\varphi + \frac{\varepsilon^2}{2} R(y,\varepsilon,\varphi)$$

  for some remainder term R such that $R(y,\varepsilon,\varphi) \to 0$ as $\varepsilon \to 0$.
- Since $y(x)$ is a point of a local minimum, this implies that

$$0 \leq J(y+\varepsilon\varphi) - J(y) = \frac{\varepsilon^2}{2} J''(y)\varphi + \frac{\varepsilon^2}{2} R(y,\varepsilon,\varphi)$$

  for all small enough ε. Letting $\varepsilon \to 0$, we conclude that

$$J''(y)\varphi = \lim_{\varepsilon\to 0}\Big(J''(y)\varphi + R(y,\varepsilon,\varphi)\Big) \geq 0.$$

Legendre condition

Theorem 4. Legendre condition

If the functional $J(y)$ attains a local minimum at the function $y(x)$, then one has $L_{y'y'}(x, y, y') \geq 0$ throughout the interval $[a, b]$.

- As a simple example, consider the functional

$$J(y) = \int_{-1}^{1} x\sqrt{1 + y'(x)^2}\, dx.$$

- In this case, the Lagrangian L is easily seen to satisfy

$$L_{y'} = \frac{xy'(x)}{\sqrt{1 + y'(x)^2}} \quad \Longrightarrow \quad L_{y'y'} = \frac{x}{(1 + y'(x)^2)^{3/2}}.$$

- Since $L_{y'y'}$ changes sign on the given interval, one finds that $J(y)$ has neither a local minimum nor a local maximum.

Legendre condition: Sketch of proof, page 1

- If it happens that $L_{y'y'} < 0$ at some point, then $L_{y'y'} < 0$ on some interval $[x_0 - \varepsilon, x_0 + \varepsilon]$ by continuity. Consider the test function

$$\varphi(x) = \left\{ \begin{array}{cc} \sin^3\left(\frac{\pi(x-x_0)}{\varepsilon}\right) & \text{if } |x - x_0| \le \varepsilon \\ 0 & \text{otherwise} \end{array} \right\}.$$

- This function is bounded for all $\varepsilon > 0$, but its derivative becomes unbounded as $\varepsilon \to 0$. One thus expects the second variation

$$J''(y)\varphi = \int_a^b \Big(L_{yy}\varphi^2 + 2L_{yy'}\varphi\varphi' + L_{y'y'}(\varphi')^2 \Big)\, dx$$

  to become negative as $\varepsilon \to 0$. This contradicts our previous theorem which asserts that $J''(y)\varphi \ge 0$ at a point of a local minimum.
- It remains to show that $J''(y)\varphi < 0$ for all small enough $\varepsilon > 0$.

Legendre condition: Sketch of proof, page 2

- Since the Lagrangian is twice continuously differentiable, one has

$$\begin{aligned} J''(y)\varphi &= \int_{x_0-\varepsilon}^{x_0+\varepsilon} \Big(L_{yy}\varphi^2 + 2L_{yy'}\varphi\varphi' + L_{y'y'}(\varphi')^2 \Big)\, dx \\ &\leq \int_{x_0-\varepsilon}^{x_0+\varepsilon} \Big(C_1\varphi^2 + C_2|\varphi\varphi'| - C_3(\varphi')^2 \Big)\, dx \end{aligned}$$

for some constants $C_1, C_2, C_3 > 0$. Letting $u = \frac{\pi}{\varepsilon}(x - x_0)$, we get

$$\begin{aligned} J''(y)\varphi &\leq \frac{\varepsilon}{\pi} \int_{-\pi}^{\pi} \left(C_1 + \frac{3\pi C_2}{\varepsilon} - \frac{9\pi^2 C_3}{\varepsilon^2} \sin^4 u \, \cos^2 u \right)\, du \\ &= 2\varepsilon C_1 + 6\pi C_2 - \frac{C_4}{\varepsilon} \end{aligned}$$

for some constants $C_1, C_2, C_4 > 0$ and the result now follows.

Poincaré inequality

- Suppose that φ is a test function on the interval $[a, b]$. Then

$$|\varphi(x)| = |\varphi(x) - \varphi(a)| \leq \int_a^x |\varphi'(y)|\, dy,$$

so one may use the Cauchy-Schwarz inequality to find that

$$\varphi(x)^2 \leq \int_a^x dy \int_a^b \varphi'(y)^2\, dy = (x - a) \int_a^b \varphi'(x)^2\, dx.$$

- Integrating over $[a, b]$, we thus obtain the Poincaré inequality

$$\int_a^b \varphi(x)^2\, dx \leq \frac{(b-a)^2}{2} \int_a^b \varphi'(x)^2\, dx.$$

Sufficient condition

Theorem 5. Sufficient condition

Suppose that $y(x)$ is a critical point of

$$J(y) = \int_a^b L(x, y(x), y'(x))\, dx$$

subject to the boundary conditions $y(a) = y_0$ and $y(b) = y_1$. Suppose also that there exists some constant $\delta > 0$ such that

$$J''(y)\varphi \geq \delta \int_a^b \varphi'(x)^2\, dx$$

for all test functions φ. Then $J(y)$ attains a local minimum at $y(x)$.

Sufficient condition: Sketch of proof, page 1

- We use Taylor's theorem with remainder to write

$$J(y+\varepsilon\varphi) = J(y) + \frac{\varepsilon^2}{2}\int_a^b \Big(L_{yy}\varphi^2 + 2L_{yy'}\varphi\varphi' + L_{y'y'}(\varphi')^2 \Big)\, dx$$

with the second derivatives of L evaluated at a point of the form

$$(x, y+t\varepsilon\varphi, y'+t\varepsilon\varphi'), \qquad 0 \le t \le 1.$$

- Since L is twice continuously differentiable, this implies that

$$\begin{aligned} J(y+\varepsilon\varphi) = J(y) &+ \frac{\varepsilon^2}{2} J''(y)\varphi \\ &+ \frac{\varepsilon^2}{2}\int_a^b \Big(R_1\varphi^2 + 2R_2\varphi\varphi' + R_3(\varphi')^2 \Big)\, dx \end{aligned}$$

for some functions R_1, R_2, R_3 which approach zero as $\varepsilon \to 0$. We now estimate the integral that appears on the right hand side.

- Let us denote the last integral by I. We then have

$$|I| \leq \int_a^b \Big(|R_1| + |R_2|\Big) \varphi(x)^2 \, dx + \int_a^b \Big(|R_2| + |R_3|\Big) \varphi'(x)^2 \, dx$$

  and we can use the Poincaré inequality to conclude that

$$|I| \leq \int_a^b R(\varepsilon, x) \cdot \varphi'(x)^2 \, dx$$

  for some positive function R which approaches zero as $\varepsilon \to 0$.

- In view of our assumption on $J''(y)\varphi$, this implies that

$$J(y + \varepsilon\varphi) - J(y) \geq \frac{\varepsilon^2}{2} \int_a^b (\delta - R(\varepsilon, x)) \cdot \varphi'(x)^2 \, dx \geq 0$$

  for all small enough ε, so $J(y)$ attains a local minimum at $y(x)$.

Sufficient condition: Example

- Consider the shortest path example that corresponds to the case

$$J(y) = \int_a^b \sqrt{1 + y'(x)^2}\, dx.$$

- In this case, the critical points are lines, namely functions $y(x)$ whose derivative is constant, say $y'(x) = c$. One can easily check that

$$L_{y'y'} = \frac{1}{(1 + y'(x)^2)^{3/2}} = \frac{1}{(1 + c^2)^{3/2}} = \delta$$

for some constant $\delta > 0$, while $L_{yy} = L_{yy'} = 0$. This implies that

$$J''(y)\varphi = \int_a^b \delta\varphi'(x)^2\, dx,$$

so the sufficient condition is satisfied and $J(y)$ has a local minimum.

Invariance

- If there is a transformation $(x, y) \to (x_*, y_*)$ such that

$$\int_a^b L(x, y, y')\, dx = \int_{a_*}^{b_*} L(x_*, y_*, y'_*)\, dx_* \quad \text{for all } a < b,$$

  we say that $J(y)$ is invariant under the given transformation.
- A very common example is time invariance

$$x_* = x + \varepsilon, \qquad y_* = y.$$

  This case arises whenever L is independent of x, for instance.
- Another common example is translation invariance

$$x_* = x, \qquad y_* = y + \varepsilon.$$

  This case arises whenever L is independent of y, for instance.

Theorem 6. Noether's theorem

Suppose $J(y)$ is invariant under a family of transformations

$$(x, y) \longrightarrow (x_*, y_*) = (f(x, y, \varepsilon), g(x, y, \varepsilon))$$

such that $x_* = x$ and $y_* = y$ when $\varepsilon = 0$. Then the quantity

$$Q = \alpha \left(L - y' L_{y'}\right) + \beta L_{y'}$$

is independent of x whenever $y(x)$ is a critical point of $J(y)$ and

$$\alpha = \left.\frac{\partial x_*}{\partial \varepsilon}\right|_{\varepsilon=0}, \qquad \beta = \left.\frac{\partial y_*}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

- The Beltrami identity is a very special case of this theorem.

Noether's theorem: Sketch of proof, page 1

- We simplify the invariance condition using a Taylor series expansion and keeping the linear terms only. This gives the identity

$$\begin{aligned} L(x_*, y_*, y_*') = L(x, y, y') &+ (x_* - x) L_x(x, y, y') \\ &+ (y_* - y) L_y(x, y, y') + (y_*' - y') L_{y'}(x, y, y'). \end{aligned}$$

- Let us express this identity in the more compact form

$$L_* = L + \Delta x \cdot L_x + \Delta y \cdot L_y + \Delta y' \cdot L_{y'}.$$

- Keeping linear terms as before, we get $\Delta x = x_* - x = \alpha\varepsilon$ and

$$\begin{aligned} \Delta y = y_*(x) - y(x) &= y_*(x) - y_*(x_*) + y_*(x_*) - y(x) \\ &= -y'(x)\Delta x + \beta\varepsilon = (\beta - \alpha y')\varepsilon. \end{aligned}$$

Noether's theorem: Sketch of proof, page 2

- We now integrate the identity above. Since $x_* = x + \alpha\varepsilon$, we have

$$\begin{aligned}
\int_a^b L\,dx &= \int_{a_*}^{b_*} L_*\,dx_* \\
&= \int_{a_*}^{b_*} \left(L + \Delta x \cdot L_x + \Delta y \cdot L_y + \Delta y' \cdot L_{y'}\right)\,dx_* \\
&= \int_a^b \left(L + \alpha\varepsilon \cdot L_x + \Delta y \cdot L_y + \Delta y' \cdot L_{y'} + \alpha'\varepsilon \cdot L\right)\,dx.
\end{aligned}$$

- Rearranging terms and integrating by parts, we conclude that

$$0 = \int_a^b (\alpha\varepsilon \cdot L)'\,dx + \Big[\Delta y \cdot L_{y'}\Big]_a^b + \int_a^b \Delta y \left(L_y - \frac{d}{dx} L_{y'}\right)\,dx.$$

- Here, the rightmost integral is zero by the Euler-Lagrange equation.

Noether's theorem: Sketch of proof, page 3

- In view of our computation above, we must thus have

$$0 = \Big[\alpha\varepsilon \cdot L + \Delta y \cdot L_{y'}\Big]_a^b$$
$$= \Big[\alpha\varepsilon \cdot L + (\beta - \alpha y')\varepsilon \cdot L_{y'}\Big]_a^b.$$

- Since the endpoints a, b are arbitrary, this actually means that

$$Q = \alpha L + (\beta - \alpha y')L_{y'}$$

is independent of x. Rearranging terms, we conclude that

$$Q = \alpha(L - y'L_{y'}) + \beta L_{y'}$$

is independent of x. This finally completes the proof.