Part I. Explicit methods

Lecture notes for MA342H

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Terminology

- **Differential equation**: An equation that relates an unknown function and its derivatives. If the unknown u depends on one variable, then it is called an ordinary differential equation (ODE) and one uses primes to denote derivatives such as u''. If the unknown u depends on two or more variables, then it is called a partial differential equation (PDE) and one uses subscripts to denote derivatives such as u_x.
- **Order**: The order of a differential equation is the order of the highest derivative that appears in the equation. For instance, $u'' + u = 0$ is a second-order ODE, while $u_x + u_y = 1$ is a first-order PDE.
- **Linear**: We say that a differential equation is linear, if the coefficients of the unknown u and its derivatives do not depend on either u or its derivatives. For instance, $e^t u' + t^3 u = t^2$ is a first-order linear ODE, while $u_t - u_{xx} = t^2 e^x u$ is a second-order linear PDE.

Cauchy-Riemann equations

- Suppose $f(z)$ is a function of a complex variable $z = x + iy$, say
$$f(z) = u(x, y) + iv(x, y)$$
with u, v both real. Then f is complex differentiable if and only if
$$u_x = v_y, \qquad u_y = -v_x. \tag{CR}$$
- These are the Cauchy-Riemann equations which also imply that
$$\begin{aligned} u_{xx} = v_{yx} = v_{xy} = -u_{yy} \quad &\implies \quad u_{xx} + u_{yy} = 0, \\ v_{xx} = -u_{yx} = -u_{xy} = -v_{yy} \quad &\implies \quad v_{xx} + v_{yy} = 0. \end{aligned}$$
- In other words, both the real part u and the imaginary part v of a complex differentiable function must satisfy the Laplace equation
$$w_{xx} + w_{yy} = 0. \tag{LE}$$
- Solutions of the Laplace equation are often called harmonic.

Transport equation

- Consider a fluid that flows through a pipe which is so thin that it may be regarded as one-dimensional. Let $u(x,t)$ and $v(x,t)$ be the density and the velocity of the fluid, respectively, at the point x at time t.
- The product $u(x,t)v(x,t)$ gives the rate at which mass is changing at the point x at time t. Letting m be the mass which has accumulated between the points $a < b$, one easily finds that

$$\begin{aligned}\int_a^b u_t(x,t)\,dx &= \frac{d}{dt}\int_a^b u(x,t)\,dx \\ &= \frac{dm}{dt} = u(a,t)v(a,t) - u(b,t)v(b,t).\end{aligned}$$

- Using the fundamental theorem of calculus, we conclude that

$$\int_a^b u_t(x,t)\,dx = -\int_a^b [u(x,t)v(x,t)]_x\,dx.$$

- As this holds for any interval (a,b), we must have $u_t + (uv)_x = 0$.

Separation of variables, page 1

- Separation of variables is a simple method for finding solutions of a special form. This method applies to a large number of linear PDE.
- As a typical example, let us consider the transport equation

$$u_t + (xtu)_x = 0 \quad \Longrightarrow \quad u_t + xtu_x + tu = 0.$$

- Looking for solutions of the form $u(x,t) = F(x)G(t)$ gives

$$F(x)G'(t) + xtF'(x)G(t) + tF(x)G(t) = 0.$$

Divide this equation by $tF(x)G(t)$ and rearrange terms to get

$$\frac{G'(t)}{tG(t)} + 1 = -\frac{xF'(x)}{F(x)}.$$

Separation of variables, page 2

- According to the last equation, we have a function of t which is equal to a function of x. Thus, each of these functions is constant, say
$$\frac{G'(t)}{tG(t)} + 1 = \lambda = -\frac{xF'(x)}{F(x)}.$$
- When it comes to the unknown $F(x)$, separation of variables gives
$$\frac{F'(x)}{F(x)} = -\frac{\lambda}{x} \quad \Longrightarrow \quad \log F(x) = -\lambda \log x + C_1$$
$$\Longrightarrow \quad F(x) = C_2 x^{-\lambda}.$$
When it comes to the unknown $G(t)$, we similarly get
$$\frac{G'(t)}{G(t)} = (\lambda - 1)t \quad \Longrightarrow \quad G(t) = C_3 e^{(\lambda-1)t^2/2}.$$
- Thus, the given PDE is satisfied by any function of the form
$$u(x,t) = F(x)G(t) = C_4 x^{-\lambda} e^{(\lambda-1)t^2/2}.$$

Method of characteristics

- This is a general method that applies to any first-order PDE in any number of variables. Let us first focus on the special case

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = c(x,y).$$

- The tangent plane to the graph of $z = u(x,y)$ at (x_0, y_0, u_0) is

$$u_x(x_0,y_0)\cdot(x-x_0) + u_y(x_0,y_0)\cdot(y-y_0) = u - u_0,$$

  so its normal vector is the vector $(u_x, u_y, -1)$. The given PDE holds if and only if the normal vector is perpendicular to (a,b,c), hence if and only if the latter vector is tangential to the graph.

- Consider the curve $(x(s), y(s), u(s))$ which is obtained by solving

$$x'(s) = a, \qquad y'(s) = b, \qquad u'(s) = c.$$

  If its initial point (x_0, y_0, u_0) lies on the graph, then the whole curve must lie on the graph. Such a curve is often called characteristic.

Characteristics: A simple example

- We use the method of characteristics to solve the equation

$$xu_x + u_y = 0, \qquad u(x,0) = f(x).$$

- In this case, the characteristic equations are given by

$$x'(s) = x, \qquad y'(s) = 1, \qquad u'(s) = 0.$$

  The initial condition describes a curve that may be written as

$$x(0) = r, \qquad y(0) = 0, \qquad u(0) = f(r).$$

- Solving this system of equations, one easily finds that

$$x(s) = x(0)e^s = re^s, \qquad y(s) = s, \qquad u(s) = u(0) = f(r).$$

  Once we now eliminate r and s, we get $u = f(xe^{-s}) = f(xe^{-y})$.

Characteristics: Quasilinear case

- A first-order quasilinear PDE has the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{Q1}$$

  We wish to solve this PDE subject to the smooth initial condition

$$u(x_0(r), y_0(r)) = u_0(r). \tag{Q2}$$

- The associated characteristic equations are given by

$$\left\{ \begin{array}{ll} x'(s) = a(x(s), y(s), u(s)), & x(0) = x_0(r) \\ y'(s) = b(x(s), y(s), u(s)), & y(0) = y_0(r) \\ u'(s) = c(x(s), y(s), u(s)), & u(0) = u_0(r) \end{array} \right\}. \tag{CE}$$

- Suppose that the coefficients a, b, c are smooth and suppose that

$$\det \begin{bmatrix} x_0'(r) & x'(0) \\ y_0'(r) & y'(0) \end{bmatrix} \neq 0.$$

  Then (Q1) – (Q2) has a unique solution around the initial curve. To obtain it, one solves (CE) and then inverts the map $(x, y) \to (r, s)$.

Solvability condition

- The solvability condition can be checked in advance because

$$\det \begin{bmatrix} x_0'(r) & x'(0) \\ y_0'(r) & y'(0) \end{bmatrix} = \det \begin{bmatrix} x_0'(r) & a(x_0(r), y_0(r), u_0(r)) \\ y_0'(r) & b(x_0(r), y_0(r), u_0(r)) \end{bmatrix}$$

  only depends on the initial curve. If this determinant happens to be zero, then the PDE may have infinitely many solutions or no solutions at all. One may handle this case by introducing another initial curve.

- The solvability condition holds if and only if the vectors

$$\begin{bmatrix} x_0'(r) \\ y_0'(r) \end{bmatrix}, \qquad \begin{bmatrix} x'(0) \\ y'(0) \end{bmatrix}$$

  are not parallel. These are precisely the directions of the initial and the characteristic curves (when projected onto the xy-plane).

Characteristics: Proof of existence

- We solve the characteristic equations subject to the initial conditions. Since the system has smooth coefficients, a unique solution exists.
- The solution x, y, u obviously depends on r, s and the Jacobian

$$\frac{\partial(x,y)}{\partial(r,s)} = \det \begin{bmatrix} x_r & x_s \\ y_r & y_s \end{bmatrix}$$

  is nonzero when $s = 0$. Thus, the Jacobian is nonzero for all small enough s and one may express each of r, s as a function of x, y.
- Now, consider $u = u(r(x,y), s(x,y))$. Since the chain rule gives

$$\begin{aligned} au_x + bu_y &= a(u_r r_x + u_s s_x) + b(u_r r_y + u_s s_y) \\ &= u_r(ar_x + br_y) + u_s(as_x + bs_y) \\ &= u_r(r_x x_s + r_y y_s) + u_s(s_x x_s + s_y y_s) \\ &= u_r r_s + u_s s_s = u_s = c, \end{aligned}$$

  this function $u = u(x,y)$ is a solution of the original PDE.

Characteristics: Proof of uniqueness

- Suppose $\widetilde{u}(x, y)$ is another solution of the PDE. To compare it with the one obtained using characteristics, consider the difference

$$w(s) = \widetilde{u}(x(s), y(s)) - u(s).$$

- According to the chain rule, we must then have

$$w'(s) = \widetilde{u}_x x' + \widetilde{u}_y y' - u' = a\widetilde{u}_x + b\widetilde{u}_y - c = 0.$$

- Thus, the difference of the two solutions is constant for all s. Since both solutions satisfy the initial condition (Q2), their difference is initially zero, so it must be zero for all s. In other words, one has

$$\widetilde{u}(x(s), y(s)) = u(s) \quad \text{for all } s.$$

Characteristics: Fully nonlinear case, page 1

- Every first-order PDE involving $u(x, y)$ can be expressed in the form

$$F(x, y, u, u_x, u_y) = 0$$

for some function F. Letting $p = u_x$ and $q = u_y$, let us now write

$$F(x, y, u, p, q) = 0.$$

- Differentiating this equation with respect to x gives

$$F_x + F_u u_x + F_p p_x + F_q q_x = 0.$$

Since $u_x = p$ and $q_x = u_{yx} = u_{xy} = p_y$, we conclude that

$$F_p p_x + F_q p_y = -F_x - pF_u.$$

- This is a first-order PDE in p with characteristic equations

$$x'(s) = F_p, \qquad y'(s) = F_q, \qquad p'(s) = -F_x - pF_u.$$

Characteristics: Fully nonlinear case, page 2

- These equations were obtained by differentiating the original PDE with respect to x. By symmetry, one must also have

$$q'(s) = -F_y - qF_u,$$

while a simple application of the chain rule gives

$$u'(s) = u_x x' + u_y y' = pF_p + qF_q.$$

- The complete system of characteristic equations is then

$$\left\{ \begin{array}{l} x'(s) = F_p \\ y'(s) = F_q \\ u'(s) = pF_p + qF_q \\ p'(s) = -F_x - pF_u \\ q'(s) = -F_y - qF_u \end{array} \right\}.$$

- One usually knows the initial values for x, y, u. Using those and the PDE, we get the initial values for p, q (which may not be unique).

Example 1. Quasilinear case

- The quasilinear case $au_x + bu_y = c$ corresponds to the case

$$ap + bq = c \quad \Longrightarrow \quad F(x, y, u, p, q) = ap + bq - c.$$

- In this case, the first 3 characteristic equations are given by

$$x'(s) = F_p = a, \qquad y'(s) = F_q = b, \qquad u'(s) = pF_p + qF_q = c.$$

  These are obviously the same equations that we had before. As the coefficients a, b, c depend on x, y, u only, we get 3 equations in 3 unknowns, so the other characteristic equations are redundant.

- In the general case, however, the first 3 characteristic equations may depend on the remaining variables p and q. Thus, one also needs to introduce the characteristic equations that involve p' and q'.

Example 2. Hamilton-Jacobi equations

- A Hamilton-Jacobi equation is an equation that has the form

$$u_t + H(x, u_x) = 0 \tag{HJ}$$

for some function H which is also known as the Hamiltonian.

- In our notation, this first-order PDE corresponds to the case

$$F(x, y, u, p, q) = F(x, t, u, p, q) = q + H(x, p).$$

- The most important characteristic equations are those given by

$$x'(s) = F_p = H_p, \qquad p'(s) = -F_x - pF_u = -H_x. \tag{HE}$$

These are called Hamilton's equations. Using them, one finds that

$$H'(s) = H_x x' + H_p p' = H_x H_p - H_p H_x = 0.$$