Wave equation: remaining facts

Theorem 1 (Duhamel's formula). The non-homogeneous problem on the real line

$$u_{tt} - c^2 u_{xx} = f(x,t), \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x)$$

has a unique solution which is given by Duhamel's formula

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,\tau)\,dy\,d\tau.$$

Letting $S(t)\psi(x)$ denote the middle term on the right hand side, one can also write

$$u(x,t) = S'(t)\varphi(x) + S(t)\psi(x) + \int_0^t S(t-\tau)f(x,\tau)\,d\tau.$$

Theorem 2 (Energy). Suppose $u(x,t)$ is a classical solution of the wave equation on the real line and suppose $u(x,t)$ vanishes as $|x| \to \infty$. Then the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t(x,t)^2 + c^2 u_x(x,t)^2\,dx$$

is conserved at all times. In fact, a similar statement holds for solutions of the associated boundary value problem on $[0,L]$ under either Dirichlet or Neumann conditions.