Wave equation on the half line

▶ **Dirichlet**: Consider the Dirichlet problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad u(0,t) = 0$$

on the half line $x > 0$. To solve this problem, one extends the initial data φ, ψ to the whole real line in such a way that the extension is odd and then solves the corresponding problem using d'Alembert's formula. This approach leads to the usual formula

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds, \qquad \text{if } x > ct,$$

while a similar formula holds in the remaining case, namely

$$u(x,t) = \frac{\varphi(ct+x) - \varphi(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(s)\,ds, \qquad \text{if } x < ct.$$

▶ **Neumann**: Consider the Neumann problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad u_x(0,t) = 0$$

on the half line $x > 0$. To solve this problem, one proceeds as above but now extends the initial data φ, ψ to the whole real line in such a way that the extension is even.

Wave equation on a closed interval

▶ **Dirichlet**: Consider the Dirichlet problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad u(0,t) = 0 = u(L,t)$$

on the closed interval $[0, L]$. Using separation of variables, one finds that

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \sin\frac{n\pi ct}{L} + b_n \cos\frac{n\pi ct}{L} \right) \cdot \sin\frac{n\pi x}{L} \tag{6.1}$$

satisfies both the PDE and the boundary conditions. To ensure that the initial conditions are also satisfied, we choose the coefficients a_n, b_n as follows. When $t = 0$, we need to have

$$\varphi(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \Longrightarrow \quad b_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L} \cdot \varphi(x)\,dx \tag{6.2}$$

by the uniqueness of Fourier coefficients, and we similarly need to have

$$\psi(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \cdot a_n \sin\frac{n\pi x}{L} \quad \Longrightarrow \quad a_n = \frac{2}{n\pi c}\int_0^L \sin\frac{n\pi x}{L} \cdot \psi(x)\,dx. \tag{6.3}$$

Thus, the solution is given by (6.1), where a_n and b_n are given by the last two equations.