Wave equation: initial value problem

Theorem 1 (Formulas). Every solution of the wave equation $u_{tt} = c^2 u_{xx}$ has the form

$$u(x,t) = F(x - ct) + G(x + ct)$$

for some functions F, G. In particular, the initial value problem on the real line

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x) \tag{5.1}$$

has a unique solution which is given by d'Alembert's formula

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds. \tag{5.2}$$

Definition 2 (Classical solutions). We say that a function is of class $\mathcal{C}^k$ or simply $\mathcal{C}^k$, if its kth-order derivatives are all continuous. A solution of a PDE is said to be classical, if it is $\mathcal{C}^k$, where k is the order of the PDE. When it comes to the Cauchy problem (5.1), a classical solution must be $\mathcal{C}^2$, so such solutions exist only when $\varphi \in \mathcal{C}^2$ and $\psi \in \mathcal{C}^1$.

Definition 3 (L^∞ norm). Given a bounded function f on $\mathbb{R}^n$, we define its L^∞ norm by

$$||f||_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

It is easy to check that this is a norm on L^∞, the space of all bounded functions on $\mathbb{R}^n$.

Definition 4 (Well-posed). A problem involving a PDE is called well-posed, if it has a unique solution and if that solution is stable with respect to some norm.

Theorem 5 (Stability for the wave equation). Let u_1 be the unique solution of the Cauchy problem (5.1) with initial data φ_1, ψ_1 and let u_2 be the unique solution with initial data φ_2, ψ_2. Then the difference of these two solutions satisfies the estimate

$$|u_1(x,t) - u_2(x,t)| \leq ||\varphi_1 - \varphi_2||_\infty + t \cdot ||\psi_1 - \psi_2||_\infty$$

at all points (x,t). Thus, the Cauchy problem (5.1) is well-posed on $[0,T]$ for any $T > 0$.