Separation of variables

Example 1. Focusing on the equation $u_t = 2u_x$, we find all solutions of the form

$$u(x,t) = F(x)G(t).$$

To say that such a function is a solution is to say that

$$u_t = 2u_x \quad \iff \quad F(x)G'(t) = 2F'(x)G(t) \quad \iff \quad \frac{G'(t)}{2G(t)} = \frac{F'(x)}{F(x)}.$$

Now, the rightmost equation requires a function of t to be equal to a function of x, and this can only happen when both functions are constant. In other words, we have

$$u_t = 2u_x \quad \iff \quad \frac{G'(t)}{2G(t)} = \frac{F'(x)}{F(x)} = \lambda,$$

where λ is an arbitrary constant. This gives a system of two ODEs and we get

$$\begin{aligned} G'(t) = 2\lambda G(t) \quad &\implies \quad G(t) = C_1 e^{2\lambda t}, \\ F'(x) = \lambda F(x) \quad &\implies \quad F(x) = C_2 e^{\lambda x}. \end{aligned}$$

In particular, the given PDE has solutions of the form $u(x,t) = F(x)G(t) = Ce^{\lambda(x+2t)}$.