Proofs of the main results

1. Homework 2, Problem 3 is quite similar. The main idea is to factor the PDE as

$$0 = (\partial_t^2 - c^2\partial_x^2)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u.$$

If we can find variables v, w such that $\partial_v = \partial_t + c\partial_x$ and $\partial_w = \partial_t - c\partial_x$, then

$$0 = \partial_v\partial_w u \quad \Longrightarrow \quad \partial_w u = F_1(w) \quad \Longrightarrow \quad u = F_2(w) + F_3(v).$$

To actually find these variables v and w, we note that

$$\partial_v = t_v\partial_t + x_v\partial_x, \qquad \partial_w = t_w\partial_t + x_w\partial_x$$

by the chain rule, while $\partial_v = \partial_t + c\partial_x$ and $\partial_w = \partial_t - c\partial_x$ by above. This gives

$$t_v = t_w = 1, \qquad x_v = c, \qquad x_w = -c$$

so we can let $t = v + w$ and $x = cv - cw$. Solving for v and w, we conclude that

$$x + ct = 2cv, \qquad x - ct = -2cw \quad \Longrightarrow \quad u = F(x - ct) + G(x + ct).$$

2. We wish to find the unique solution of the initial value problem

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x).$$

Using the general form for solutions of the wave equation, we then get

$$u(x,t) = F(x + ct) + G(x - ct), \qquad u_t(x,t) = cF'(x + ct) - cG'(x - ct)$$

for some functions F, G. To ensure that the initial conditions hold, we need to have

$$\varphi(x) = F(x) + G(x), \qquad \psi(x) = cF'(x) - cG'(x).$$

Integrating the rightmost equation gives the equivalent system

$$F(x) + G(x) = \varphi(x), \qquad F(x) - G(x) = \frac{1}{c}\int_0^x \psi(s)\,ds.$$

Adding and subtracting these two equations, it is then easy to see that

$$F(x) = \frac{\varphi(x)}{2} + \frac{1}{2c}\int_0^x \psi(s)\,ds, \qquad G(x) = \frac{\varphi(x)}{2} - \frac{1}{2c}\int_0^x \psi(s)\,ds.$$

In particular, the unique solution of the initial value problem is given by

$$u(x,t) = F(x + ct) + G(x - ct) = \frac{\varphi(x + ct) + \varphi(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

3. Those are four different problems. See Homework 2, Problem 1 for the wave equation with even initial data and Homework 3, Problem 7 for the heat equation with even initial data; the remaining two problems are quite similar.

4. Those are four different problems. See Homework 2, Problem 2 for the wave equation with Neumann boundary conditions and Homework 3, Problem 2 for the heat equation with Dirichlet boundary conditions; the remaining two problems are quite similar.

5. Those are two different problems, but they are almost identical. Define the energy by

$$E(t) = \frac{1}{2}\int_0^L u_t(x,t)^2 + c^2 u_x(x,t)^2\,dx = \frac{1}{2}\int_0^L u_t^2 + c^2 u_x^2\,dx.$$

To show this is conserved for solutions of the wave equation $u_{tt} = c^2 u_{xx}$, note that

$$E'(t) = \int_0^L u_t u_{tt} + c^2 u_x u_{xt}\,dx = \int_0^L c^2 u_t u_{xx} + c^2 u_x u_{xt}\,dx.$$

Integrating one of the two integrals by parts, we now get

$$\begin{aligned} E'(t) &= \Big[c^2 u_t u_x\Big]_{x=0}^{L} - \int_0^L c^2 u_{xt} u_x\,dx + \int_0^L c^2 u_x u_{xt}\,dx \\ &= \Big[c^2 u_t u_x\Big]_{x=0}^{L}. \end{aligned}$$

For the Neumann problem, $u_x = 0$ on the boundary and we get $E'(t) = 0$, indeed. For the Dirichlet problem, $u = 0$ on the boundary at all times, so $u_t = 0$ on the boundary and the same conclusion holds.

6. Suppose that u_1, u_2 are both solutions of the boundary value problem

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x)$$

on $[0, L]$ subject to either Dirichlet or Neumann conditions. Then $w = u_1 - u_2$ satisfies

$$w_{tt} = c^2 w_{xx}, \qquad w(x,0) = w_t(x,0) = 0$$

subject to the same boundary conditions. Using conservation of energy, we now get

$$\int_0^L w_t(x,t)^2 + c^2 w_x(x,t)^2\,dx = \int_0^L w_t(x,0)^2 + c^2 w_x(x,0)^2\,dx.$$

Since both w_t and w_x are initially zero by above, the last equation implies

$$\int_0^L w_t(x,t)^2 + c^2 w_x(x,t)^2\,dx = 0 \quad\Longrightarrow\quad w_t(x,t) = w_x(x,t) = 0$$

at all times. This means that $w(x,t)$ is constant; being initially zero, it must thus be zero at all times and so u_1, u_2 are identical.

7. Let u_1 be the solution of the wave equation when the initial data φ_1, ψ_1 are imposed and let u_2 be the solution for the initial data φ_2, ψ_2. We need to show that u_1 and u_2 remain close at all times, if they are initially close. By d'Alembert's formula,

$$u_1(x,t) = \frac{\varphi_1(x+ct) + \varphi_1(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_1(s)\, ds$$

and a similar formula holds for u_2; subtracting these two formulas now gives

$$u_1(x,t) - u_2(x,t) = \frac{\varphi_1(x+ct) - \varphi_2(x+ct)}{2} + \frac{\varphi_1(x-ct) - \varphi_2(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_1(s) - \psi_2(s)\, ds.$$

Letting $||f||_\infty$ denote the L^∞ norm of a function f, we deduce that

$$\begin{aligned} |u_1(x,t) - u_2(x,t)| &\leq \frac{||\varphi_1 - \varphi_2||_\infty}{2} + \frac{||\varphi_1 - \varphi_2||_\infty}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} ||\psi_1 - \psi_2||_\infty\, ds \\ &= ||\varphi_1 - \varphi_2||_\infty + t \cdot ||\psi_1 - \psi_2||_\infty. \end{aligned}$$

To prove stability, let $\varepsilon > 0$ be given and suppose the initial data are so close that

$$||\varphi_1 - \varphi_2||_\infty \leq \frac{\varepsilon}{2}, \qquad ||\psi_1 - \psi_2||_\infty \leq \frac{\varepsilon}{2T}.$$

Combining the last two equations, we then get

$$|u_1(x,t) - u_2(x,t)| \leq \frac{\varepsilon}{2} + \frac{t\varepsilon}{2T} \leq \varepsilon \quad \Longrightarrow \quad ||u_1 - u_2||_\infty \leq \varepsilon.$$

8. **Statement**. Suppose u satisfies the heat equation $u_t = ku_{xx}$ in some closed, bounded region A in the xt-plane. Then both the minimum and the maximum values of u are attained on the boundary of A.

Proof. Fix some $\varepsilon > 0$ and let $v(x,t) = u(x,t) + \varepsilon x^2$. If v attains its maximum at an interior point, then $v_t = 0$ and $v_{xx} \leq 0$ at that point, so $v_t - kv_{xx} \geq 0$ there. Since

$$v_t - kv_{xx} = (u_t - ku_{xx}) - 2k\varepsilon = -2k\varepsilon < 0$$

at all points, however, the maximum of v is attained on the boundary. Using the fact that A is a bounded region, we now find

$$v(x,t) = u(x,t) + \varepsilon x^2 \leq \max_{\partial A} u + C\varepsilon$$

at all points on the boundary, hence also at all points. Letting $\varepsilon \to 0$, this gives

$$u(x,t) \leq \max_{\partial A} u$$

at all points, so the maximum of u is also attained on the boundary. Since

$$\min u = -\max(-u)$$

and $-u$ itself satisfies the heat equation, $\min u$ is attained on the boundary as well.

9. First of all, let us invoke the explicit formula for the solution, which gives

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t) \cdot u(y,0)\, dy.$$

Since the heat kernel $S(x,t)$ is non-negative, we then have

$$\begin{aligned} |u(x,t)| &\leq \int_{-\infty}^{\infty} S(x-y,t) \cdot |u(y,0)|\, dy \\ &= \int_{-\infty}^{\infty} S(x-y,t)^{\frac{1}{q}} \cdot S(x-y,t)^{\frac{1}{p}}\, |u(y,0)|\, dy \end{aligned}$$

whenever $\frac{1}{p} + \frac{1}{q} = 1$. Using Hölder's inequality to estimate the integral, we get

$$|u(x,t)| \leq \left[\int_{-\infty}^{\infty} S(x-y,t)^{\frac{1}{q}\cdot q}\, dy \right]^{1/q} \left[\int_{-\infty}^{\infty} S(x-y,t)^{\frac{1}{p}\cdot p}\, |u(y,0)|^p\, dy \right]^{1/p}$$

and the expression in the first pair of brackets is equal to 1, so

$$|u(x,t)|^p \leq \int_{-\infty}^{\infty} S(x-y,t)\, |u(y,0)|^p\, dy.$$

Integrating with respect to x and using Fubini's theorem, we conclude that

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x,t)|^p\, dx &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(x-y,t)\, |u(y,0)|^p\, dy\, dx \\ &= \int_{-\infty}^{\infty} |u(y,0)|^p \int_{-\infty}^{\infty} S(x-y,t)\, dx\, dy \\ &= \int_{-\infty}^{\infty} |u(y,0)|^p\, dy. \end{aligned}$$

10. **Statement**. The average value of a harmonic function u over a sphere is equal to its value at the centre. In other words, one has

$$u(x) = \frac{1}{n\alpha_n r^{n-1}} \int_{|y-x|=r} u(y)\, dS_y$$

for all $x \in \mathbb{R}^n$ and all $r > 0$, where α_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. Suppose $u(x)$ is harmonic and let $I(x,r)$ be its mean value over the sphere of radius r around the point $x \in \mathbb{R}^n$. Then we have

$$I(x,r) = \frac{1}{n\alpha_n r^{n-1}} \int_{|y-x|=r} u(y)\, dS_y$$

and we can change variables by $z = \frac{y-x}{r}$ to get

$$I(x,r) = \frac{1}{n\alpha_n r^{n-1}} \int_{|z|=1} u(x+rz)\, r^{n-1} dS_z = \frac{1}{n\alpha_n} \int_{|z|=1} u(x+rz)\, dS_z.$$

Differentiating this expression with respect to r, one finds that

$$I_r(x,r) = \frac{1}{n\alpha_n} \int_{|z|=1} \nabla u(x+rz) \cdot z\, dS_z = \frac{1}{n\alpha_n} \int_{|z|=1} \nabla u(x+rz) \cdot \boldsymbol{n}\, dS_z,$$

where $\boldsymbol{n}$ is the outward unit normal vector on the sphere $|z| = 1$. Using the divergence theorem together with the fact that u is harmonic, we thus get

$$I_r(x,r) = \frac{1}{n\alpha_n} \int_{|z|\leq 1} \nabla \cdot \nabla u(x+rz)\, dz = \frac{1}{n\alpha_n} \int_{|z|\leq 1} \Delta u(x+rz)\, dz = 0.$$

This shows that the mean value $I(x,r)$ is independent of the radius r of the sphere. Letting $r \to 0$ makes the sphere shrink down to a point, so it easily follows that

$$I(x,r) = \lim_{r\to 0} I(x,r) = u(x).$$

11. **Statement**. If a function u is harmonic in some closed, bounded region $A \subset \mathbb{R}^n$, then both the min and the max values of u are attained on the boundary.

Proof. Suppose the maximum M is attained at an interior point x. Let $r > 0$ be the distance of x from the boundary and let $S \subset A$ be the sphere of radius r around x. By the mean value property for harmonic functions over spheres, we must then have

$$M = u(x) = \frac{1}{n\alpha_n r^{n-1}} \int_S u(y)\, dS_y \leq \frac{1}{n\alpha_n r^{n-1}} \int_S M\, dS_y = M.$$

Thus, equality holds in the inequality above and $u(y) = M$ at all points on the sphere. Since that includes points on the boundary, the maximum is attained there. Since

$$\min u = -\max(-u)$$

and $-u$ is harmonic as well, the minimum of u is also attained on the boundary.

12. Pick any function $G(x)$ which is non-negative, radial and smooth with

$$\int_{\mathbb{R}^n} G(x)\, dx = 1.$$

To see that such a function exists, let $S(x,t)$ be the heat kernel and define

$$G(x) = S(x_1,1) \cdot S(x_2,1) \cdot \ldots \cdot S(x_n,1) = \frac{1}{(4k\pi)^{n/2}} \cdot \exp\left(-\frac{|x|^2}{4k}\right).$$

We claim that $u(x)$ is equal to the convolution $G(x) * u(x)$. In fact, we have

$$\begin{aligned} G(x) * u(x) &= \int_{\mathbb{R}^n} G(x-y)u(y)\,dy \\ &= \int_0^\infty \int_{|x-y|=r} G(r)u(y)\,dS_y\,dr \end{aligned}$$

because G is radial, so the mean value property for harmonic functions gives

$$G(x) * u(x) = u(x) \int_0^\infty G(r) \cdot n\alpha_n r^{n-1}\,dr,$$

where α_n is the volume of the unit ball in $\mathbb{R}^n$. Since the last integral is equal to

$$\int_0^\infty G(r) \cdot n\alpha_n r^{n-1}\,dr = \int_0^\infty \int_{|z|=r} G(z)\,dS_z\,dr = \int_{\mathbb{R}^n} G(x)\,dx = 1$$

by assumption, we have actually shown that

$$u(x) = G(x) * u(x) = \int_{\mathbb{R}^n} G(x-y)u(y)\,dy.$$

Using this equation and the fact that G is smooth, we find that u is also smooth.

13. **Statement**. If $u\colon \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^1$ and $v\colon \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^2$, then one has

$$\int_A u\Delta v\,dx = -\int_A \nabla u \cdot \nabla v\,dx + \int_{\partial A} u\nabla v \cdot \boldsymbol{n}\,dS.$$

If u, v are both $\mathcal{C}^2$, then one similarly has

$$\int_A (u\Delta v - v\Delta u)\,dx = \int_{\partial A} (u\nabla v - v\nabla u) \cdot \boldsymbol{n}\,dS.$$

Proof. According to the product rule, the divergence of $u\nabla v = \langle uv_{x_1}, \ldots, uv_{x_n}\rangle$ is

$$\nabla \cdot (u\nabla v) = \sum_{i=1}^n (uv_{x_i})_{x_i} = \sum_{i=1}^n u_{x_i}v_{x_i} + uv_{x_i x_i} = \nabla u \cdot \nabla v + u\Delta v.$$

We now integrate over A and apply the divergence theorem to get

$$\int_{\partial A} u\nabla v \cdot \boldsymbol{n}\,dS = \int_A \nabla u \cdot \nabla v\,dx + \int_A u\Delta v\,dx.$$

This is Green's first identity. Interchanging the roles of u, v also gives

$$\int_{\partial A} v\nabla u \cdot \boldsymbol{n}\,dS = \int_A \nabla u \cdot \nabla v\,dx + \int_A v\Delta u\,dx,$$

so we may subtract the last two equations to obtain Green's second identity.

14. There are two different ways to prove uniqueness; see Homework 4, Problem 8.

15. Suppose u is harmonic and bounded. Then $\Delta u = 0$ and u is smooth, so

$$\Delta u_{x_i} = 0 \quad \Longrightarrow \quad u_{x_i}(x) = \frac{1}{\alpha_n r^n} \int_{|y-x|\leq r} u_{x_i}(y)\, dy$$

by the mean value property for harmonic functions. Now, u_{x_i} is the divergence of the vector $\boldsymbol{F}$ whose ith entry is equal to u, all other entries being zero. This gives

$$u_{x_i}(x) = \frac{1}{\alpha_n r^n} \int_{|y-x|\leq r} \nabla \cdot \boldsymbol{F}(y)\, dy = \frac{1}{\alpha_n r^n} \int_{|y-x|=r} \boldsymbol{F}(y) \cdot \boldsymbol{n}\, dS_y$$

because of the divergence theorem. Since $\boldsymbol{n}$ is a unit vector and u bounded, we get

$$|u_{x_i}(x)| \leq \frac{1}{\alpha_n r^n} \int_{|y-x|=r} ||u||_\infty \, dS_y = \frac{||u||_\infty}{\alpha_n r^n} \cdot n\alpha_n r^{n-1} = \frac{n||u||_\infty}{r}\,.$$

This inequality holds for any $r > 0$, so we can let $r \to \infty$ to find that u_{x_i} is zero at all points. In particular, u is independent of each x_i and must thus be constant.

16. We have to prove the identity

$$-\int_{\mathbb{R}^3} F(x)\Delta\varphi(x)\, dx = \varphi(0)$$

for all test functions φ. Let us then fix some $\varepsilon > 0$ and write

$$-\int_{\mathbb{R}^3} F(x)\Delta\varphi(x)\, dx = -\int_{|x|\leq\varepsilon} F(x)\Delta\varphi(x)\, dx - \int_{|x|\geq\varepsilon} F(x)\Delta\varphi(x)\, dx.$$

Since F is harmonic in the region $|x| \geq \varepsilon$, Green's second identity gives

$$-\int_{|x|\geq\varepsilon} F\Delta\varphi\, dx = \int_{|x|=\varepsilon} (\varphi\nabla F - F\nabla\varphi) \cdot \boldsymbol{n}\, dS_x,$$

where $\boldsymbol{n} = -x/\varepsilon$ is the outward unit normal vector. This allows us to write

$$\begin{aligned} -\int_{\mathbb{R}^3} F\Delta\varphi\, dx &= -\int_{|x|\leq\varepsilon} F\Delta\varphi\, dx + \int_{|x|=\varepsilon} \varphi\nabla F \cdot \boldsymbol{n}\, dS_x - \int_{|x|=\varepsilon} F\nabla\varphi \cdot \boldsymbol{n}\, dS_x \\ &= I_1 + I_2 + I_3 \end{aligned}$$

as the sum of three integrals. When it comes to the first integral, we have

$$|I_1| \leq \int_{|x|\leq\varepsilon} \frac{||\Delta\varphi||_\infty}{4\pi|x|}\, dx = C\int_0^\varepsilon \int_{|x|=\rho} \rho^{-1} dS_x\, d\rho$$

and the surface of the sphere $|x| = \rho$ is equal to $4\pi\rho^2$, so

$$|I_1| \le C \int_0^\varepsilon \rho\, d\rho = C\varepsilon^2$$

and $I_1 \to 0$ as $\varepsilon \to 0$. When it comes to the third integral, we similarly get

$$|I_3| \le \int_{|x|=\varepsilon} \frac{||\nabla\varphi||_\infty}{4\pi|x|}\, dS_x = C \int_{|x|=\varepsilon} \varepsilon^{-1} dS_x = C\varepsilon$$

so $I_3 \to 0$ as $\varepsilon \to 0$. Once we now combine these observations, we arrive at

$$-\int_{\mathbb{R}^3} F\Delta\varphi\, dx = \lim_{\varepsilon\to 0} \int_{|x|=\varepsilon} \varphi\nabla F \cdot \boldsymbol{n}\, dS_x.$$

Using the explicit formulas for $F(x)$ and $\boldsymbol{n} = -x/\varepsilon$, it is easy to check that

$$\nabla F(x) \cdot \boldsymbol{n} = \frac{1}{4\pi\varepsilon|x|}\,.$$

Since the sphere $|x| = \varepsilon$ has surface $4\pi\varepsilon^2$, we may finally conclude that

$$-\int_{\mathbb{R}^3} F\Delta\varphi\, dx = \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon^2} \int_{|x|=\varepsilon} \varphi(x)\, dS_x = \varphi(0).$$

17. **Statement**. Suppose $A \subset \mathbb{R}^n$ is bounded and $f\colon \mathbb{R}^n \to \mathbb{R}$ a given function. Out of all functions that satisfy $u(x) = f(x)$ on ∂A, the one that minimizes

$$I(u) = \int_A |\nabla u|^2\, dx$$

is the one which is harmonic within A.

Proof. Suppose $u(x) = v(x)$ on the boundary and let $w = u - v$. Then we have

$$\begin{aligned} I(u) &= \int_A |\nabla v + \nabla w|^2\, dx = \int_A (\nabla v + \nabla w) \cdot (\nabla v + \nabla w)\, dx \\ &= \int_A |\nabla v|^2 + 2\nabla v \cdot \nabla w + |\nabla w|^2\, dx. \end{aligned}$$

Using Green's identity and the fact that $w = 0$ on the boundary, we also get

$$\int_A \nabla v \cdot \nabla w\, dx = -\int_A w\Delta v\, dx + \int_{\partial A} w\nabla v \cdot \boldsymbol{n}\, dS = -\int_A w\Delta v\, dx.$$

If we now assume that v is harmonic, then this integral vanishes and so

$$I(u) = \int_A |\nabla v|^2 + |\nabla w|^2\, dx \ge \int_A |\nabla v|^2\, dx = I(v).$$

18. The proof is similar to that of the mean value property. First, we write

$$U(x,t,r) = \frac{1}{n\alpha_n r^{n-1}} \int_{|y-x|=r} u(y,t)\,dS_y = \frac{1}{n\alpha_n} \int_{|z|=1} u(x+rz,t)\,dS_z$$

using the substitution $z = \frac{y-x}{r}$, and then we differentiate to get

$$\begin{aligned} U_r(x,t,r) &= \frac{1}{n\alpha_n} \int_{|z|=1} \nabla u(x+rz,t)\cdot z\,dS_z \\ &= \frac{1}{n\alpha_n r^{n-1}} \int_{|y-x|=r} \nabla u(y,t)\cdot \boldsymbol{n}\,dS_y, \end{aligned}$$

where $\boldsymbol{n} = z$ is the outward unit normal vector on the sphere $|y-x| = r$. Using the divergence theorem and the fact that u satisfies the wave equation, we now find

$$\begin{aligned} U_r(x,t,r) &= \frac{1}{n\alpha_n r^{n-1}} \int_{|y-x|\leq r} \Delta u(y,t)\,dy \\ &= \frac{1}{n\alpha_n r^{n-1} c^2} \int_{|y-x|\leq r} u_{tt}(y,t)\,dy. \end{aligned}$$

Next, we switch to polar coordinates and we write

$$r^{n-1} U_r(x,t,r) = \frac{1}{n\alpha_n c^2} \int_0^r \int_{|y-x|=\rho} u_{tt}(y,t)\,dS_y\,d\rho.$$

Differentiating with respect to r and using the definition of U, we arrive at

$$(n-1)r^{n-2}U_r + r^{n-1}U_{rr} = \frac{1}{n\alpha_n c^2} \int_{|y-x|=r} u_{tt}(y,t)\,dS_y = \frac{r^{n-1}U_{tt}}{c^2}.$$

Once we now solve for U_{tt}, we deduce the desired identity, namely

$$U_{tt} = c^2 \left(\frac{n-1}{r} \cdot U_r + U_{rr} \right).$$

19. Since u is a weak solution of $u_t + F(u)_x = 0$, it must satisfy the equation

$$\int_0^\infty \int_{-\infty}^\infty [u\varphi_t + F(u)\varphi_x]\,dx\,dt = 0 \tag{1}$$

for any test function φ which vanishes when $t = 0$. Now, consider the vector

$$\boldsymbol{F} = \begin{bmatrix} u\varphi \\ F(u)\varphi \end{bmatrix}$$

whose divergence is given by

$$\nabla \cdot \boldsymbol{F} = (u\varphi)_t + (F(u)\varphi)_x = [u_t + F(u)_x]\varphi + [u\varphi_t + F(u)\varphi_x].$$

Integrating over the region A^- defined by $x < h(t)$, one finds

$$\int_{\partial A^-} \boldsymbol{F} \cdot \boldsymbol{n} \, dS = \int_{A^-} [u\varphi_t + F(u)\varphi_x] \, dx \, dt + \int_{A^-} [u_t + F(u)_x]\varphi \, dx \, dt,$$

where $\boldsymbol{n} = \langle n_1, n_2 \rangle$ is the outward unit normal vector to A^-. Since u is smooth in A^- by assumption, it is a classical solution there and the rightmost integral vanishes. If we now denote by A^+ the region $x > h(t)$, then the same argument gives

$$-\int_{\partial A^+} \boldsymbol{F} \cdot \boldsymbol{n} \, dS = \int_{A^+} [u\varphi_t + F(u)\varphi_x] \, dx \, dt + \int_{A^+} [u_t + F(u)_x]\varphi \, dx \, dt$$

and the rightmost integral is zero. Using the last two equations and (1), we now get

$$\int_{x=h(t)} \Big(u^- n_1 + F(u^-) n_2 \Big) \varphi \, dS = \int_{x=h(t)} \Big(u^+ n_1 + F(u^+) n_2 \Big) \varphi \, dS$$

for all test functions φ which vanish when $t = 0$. This is easily seen to imply

$$\frac{F(u^+) - F(u^-)}{u^+ - u^-} = -\frac{n_1}{n_2} = h'(t)$$

because the vector $\boldsymbol{v} = \langle n_2, -n_1 \rangle$ is tangent to the curve $x = h(t)$.