Useful ODE facts

Definition 1 (Separable). An ODE is said to be separable, if it has the form

$$y'(x) = f(x) \cdot g(y)$$

for some functions f, g. To solve such an ODE, one simply separates variables to get

$$\frac{dy}{dx} = f(x) \cdot g(y) \quad \implies \quad \int \frac{dy}{g(y)} = \int f(x)\, dx.$$

Theorem 2 (First-order linear). To solve the first-order linear ODE

$$y'(x) + P(x)y(x) = Q(x),$$

one introduces the integrating factor $\mu(x) = \exp\left(\int P(x)\, dx\right)$ and then notes that

$$y'(x) + P(x)y(x) = Q(x) \quad \iff \quad \mu(x)y(x) = \int \mu(x)Q(x)\, dx.$$

Theorem 3 (Gronwall inequality). Suppose f, g are non-negative and continuous with

$$f(t) \leq C + \int_{t_0}^{t} f(s)g(s)\, ds$$

for some fixed constants t_0 and C. Then it must be the case that

$$f(t) \leq C \exp\left(\int_{t_0}^{t} g(s)\, ds\right).$$

Theorem 4 (Second-order ODEs). Consider the second-order ODE $y'' + by' + cy = 0$ and the associated quadratic equation $\lambda^2 + b\lambda + c = 0$, where b, c are fixed constants.

- If the quadratic has two distinct roots $\lambda_1 \neq \lambda_2$, then $y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$.
- If the quadratic has a double root λ, then $y = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$.
- If the quadratic has complex roots $\lambda = \alpha \pm \beta i$, then $y = C_1 e^{\alpha t} \sin(\beta t) + C_2 e^{\alpha t} \cos(\beta t)$.