PDEs, Homework #5
Solutions

1. *Consider the function $u\colon \mathbb{R} \to \mathbb{R}$ defined by*

$$u(x) = \left\{ \begin{array}{cc} 0 & \text{if } x < 0 \\ \sin x & \text{if } x \geq 0 \end{array} \right\}.$$

Show that $u'' + u = \delta$ in the sense of distributions. Hint: you need to show that

$$\int_{-\infty}^{\infty} u(x)\varphi''(x)\,dx + \int_{-\infty}^{\infty} u(x)\varphi(x)\,dx = \varphi(0)$$

for all test functions φ; simplify the first integral and then integrate by parts.

- Using the definition of u and an integration by parts, we find

$$\int_{-\infty}^{\infty} u(x)\varphi''(x)\,dx = \int_0^{\infty} \varphi''(x)\sin x\,dx = -\int_0^{\infty} \varphi'(x)\cos x\,dx$$

because $\varphi'(x)\sin x$ vanishes at the two endpoints. Integrating by parts again,

$$\begin{aligned} \int_{-\infty}^{\infty} u(x)\varphi''(x)\,dx &= -\int_0^{\infty} \varphi(x)\sin x\,dx - \Big[\varphi(x)\cos x\Big]_{x=0}^{\infty} \\ &= -\int_{-\infty}^{\infty} \varphi(x)u(x)\,dx + \varphi(0) \end{aligned}$$

and this implies the desired identity

$$\langle u'' + u, \varphi\rangle = \int_{-\infty}^{\infty} u(x)\varphi''(x)\,dx + \int_{-\infty}^{\infty} u(x)\varphi(x)\,dx = \varphi(0) = \langle \delta, \varphi\rangle.$$

2. *Consider the heat equation $u_t = k\Delta u$ over a bounded region $A \subset \mathbb{R}^n$ subject to zero Dirichlet boundary conditions. Show that each solution has a decreasing L^2-norm:*

$$\frac{d}{dt}\int_A u(x,t)^2\,dx \leq 0$$

at all times. Hint: differentiate, use the PDE and then use Green's identities.

- Following the hint, let us first differentiate to get

$$\frac{d}{dt}\int_A u(x,t)^2\,dx = 2\int_A uu_t\,dx = 2k\int_A u\Delta u\,dx.$$

Using Green's first identity and the fact that $u = 0$ on the boundary, we find

$$\frac{d}{dt}\int_A u(x,t)^2\,dx = -2k\int_A \nabla u \cdot \nabla u\,dx = -2k\int_A |\nabla u|^2\,dx \leq 0.$$

3. *Let $u(x,t)$ be a compactly supported solution of the wave equation $u_{tt} = \Delta u$. Show that its energy $E(t)$ is conserved, where*

$$E(t) = \int_{\mathbb{R}^n} u_t(x,t)^2 + |\nabla u(x,t)|^2 \, dx.$$

- The computation is similar to that for the one-dimensional problem. We write

$$E(t) = \int_{\mathbb{R}^n} u_t^2 + \sum_{i=1}^n u_{x_i}^2 \, dx$$

for simplicity and then we differentiate to get

$$E'(t) = \int_{\mathbb{R}^n} 2u_t u_{tt} + \sum_{i=1}^n 2u_{x_i} u_{x_i t} \, dx = \int_{\mathbb{R}^n} 2u_t \Delta u + \sum_{i=1}^n 2u_{x_i} u_{x_i t} \, dx.$$

Integrating one of these integrals by parts, we conclude that

$$E'(t) = \int_{\mathbb{R}^n} 2u_t \Delta u - \sum_{i=1}^n 2u_{x_i x_i} u_t \, dx = 0.$$

4. *Consider the initial value problem for the Burgers' equation*

$$u_t + uu_x = 0, \qquad u(x,0) = f(x) \tag{BE}$$

when $f(x) = \sin x$. Show that u is bounded at all times, whereas u_x is not.

- As usual, one has the formula $u = f(x_0) = f(x - ut)$ along characteristics, so

$$u = f(x - ut) = \sin(x - ut)$$

is certainly bounded. To see that u_x is not bounded, we note that

$$u_x = \cos(x - ut) \cdot (x - ut)_x = \cos(x - ut) \cdot (1 - tu_x)$$

by the chain rule. Solving for u_x and recalling that $x_0 = x - ut$, we then get

$$u_x = \frac{\cos(x - ut)}{1 + t\cos(x - ut)} = \frac{\cos x_0}{1 + t\cos x_0}\,.$$

If $x_0 = \pi$, for instance, then $u_x = -1/(1-t)$ becomes unbounded as $t \to 1$.

5. *Solve the initial value problem for the Burgers' equation* (BE) *when* $f(x) = -1/x$.

- First of all, we use the standard formula $u = f(x - ut)$ to get

$$u = f(x - ut) = -\frac{1}{x - ut} \quad \Longrightarrow \quad tu^2 - xu - 1 = 0.$$

This quadratic equation has two solutions which are given by

$$u = \frac{x \pm \sqrt{x^2 + 4t}}{2t}.$$

Since the denominator vanishes when $t = 0$, the numerator must also do, hence

$$x \pm \sqrt{x^2} = 0 \quad \Longrightarrow \quad \pm |x| = -x.$$

This means that the correct sign is $+$ when $x < 0$ and $-$ when $x > 0$, namely

$$u(x,t) = \left\{ \begin{array}{cl} \frac{x + \sqrt{x^2+4t}}{2t} & \text{if } x < 0 \\ & \\ \frac{x - \sqrt{x^2+4t}}{2t} & \text{if } x > 0 \end{array} \right\}.$$

In particular, the solution has a jump discontinuity along $x = 0$ because

$$\lim_{x \to 0^-} u(x,t) = \frac{\sqrt{4t}}{2t}, \qquad \lim_{x \to 0^+} u(x,t) = -\frac{\sqrt{4t}}{2t}.$$

Since the average of the two limits is zero, the Rankine-Hugoniot condition does hold and our solution is the unique weak solution of the problem.

6. *Solve the initial value problem for the Burgers' equation* (BE) *when* f *is the function defined by* $f(x) = 1$ *if* $x < 0$ *and* $f(x) = 0$ *if* $x > 0$.

- We use the standard formula $u = f(x_0) = f(x - ut)$ and distinguish two cases.

Case 1. If $x_0 < 0$, then we have $u = 1$ and also $x_0 = x - t < 0$.

Case 2. If $x_0 > 0$, then we have $u = 0$ and also $x_0 = x > 0$.

At any point with $0 < x < t$, we must thus have $1 = u = 0$, a contradiction. This shows that no classical solutions exist. Looking for a weak solution of the form

$$u(x,t) = \left\{ \begin{array}{ll} 1 & \text{if } x < h(t) \\ 0 & \text{if } x > h(t) \end{array} \right\},$$

we get $h'(t) = 1/2$ by the Rankine-Hugoniot condition. Since the curve $x = h(t)$ must pass through the point $x = 0$ when $t = 0$, it easily follows that $h(t) = t/2$.

7. *Consider the initial value problem*

$$u_t + g(u)u_x = 0, \qquad u(x,0) = f(x).$$

Show that some characteristic curves will intersect, unless $g(f(x))$ *is increasing.*

- Suppose $g(f(x))$ is not increasing. Then there exist points $x_1 < x_2$ such that

$$g(f(x_1)) > g(f(x_2)).$$

To find the characteristic curve that passes through $(x_i, 0)$, we solve the system

$$\frac{dt}{ds} = 1, \qquad \frac{dx}{ds} = g(u), \qquad \frac{du}{ds} = 0$$

subject to the initial condition $u(x_i, 0) = f(x_i)$; this gives

$$t = s, \qquad u = f(x_i), \qquad x = g(u)s + x_i = g(f(x_i))t + x_i.$$

Thus, the curve passing through $(x_1, 0)$ will intersect the one through $(x_2, 0)$ when

$$g(f(x_1))t + x_1 = g(f(x_2))t + x_2 \quad\Longrightarrow\quad t = \frac{x_2 - x_1}{g(f(x_1)) - g(f(x_2))} > 0.$$

8. *Let* $a > 0$ *be fixed. Solve the initial value problem* (BE) *in the case that*

$$f(x) = \left\{ \begin{array}{cc} a & \text{if } x \leq 0 \\ a(1-x) & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{array} \right\}.$$

- We use the standard formula $u = f(x_0) = f(x - ut)$ and distinguish three cases.

Case 1. If $x_0 \leq 0$, then we have $u = a$ and also $x_0 = x - at \leq 0$.

Case 2. If $x_0 \geq 1$, then we have $u = 0$ and also $x_0 = x \geq 1$.

Case 3. If $0 < x_0 < 1$, then we have $u = a(1 - x_0)$ and also

$$\begin{aligned} x_0 = x - at(1 - x_0) \quad &\Longrightarrow \quad 1 - x_0 = 1 - x + at(1 - x_0) \\ &\Longrightarrow \quad u = a(1 - x_0) = \frac{a(1-x)}{1 - at}. \end{aligned}$$

Putting these facts together, we find that the solution is given by

$$u(x,t) = \left\{ \begin{array}{cc} a & \text{if } x \leq at \\ \frac{a(1-x)}{1-at} & \text{if } at < x < 1 \\ 0 & \text{if } x \geq 1 \end{array} \right\}.$$

Now, this classical solution is only defined up to time $t = 1/a$. After that time, the characteristic curves start to intersect one another, so no classical solution exists. Let us then look for a weak solution of the form

$$u(x,t) = \left\{ \begin{array}{ll} a & \text{if } x < h(t) \\ 0 & \text{if } x > h(t) \end{array} \right\}.$$

Since $h'(t) = a/2$ by the Rankine-Hugoniot condition, we easily get

$$h'(t) = \frac{a}{2} \quad \Longrightarrow \quad h(t) = \frac{at}{2} + C \quad \Longrightarrow \quad h(t) = \frac{at+1}{2}$$

because the curve $x = h(t)$ must pass through the point $x = 1$ when $t = 1/a$.