PDEs, Homework #4
Solutions

1. *Show that the average value of a harmonic function over a ball is equal to its value at the centre. In other words, show that a harmonic function u satisfies*

$$u(x) = \frac{1}{\alpha_n r^n} \int_{|y-x|\le r} u(y)\,dy$$

for all $x \in \mathbb{R}^n$ and all $r > 0$, where α_n is the volume of the unit ball in $\mathbb{R}^n$. Hint: write the integral in polar coordinates and use the mean value property for spheres.

- Let $I(x,r)$ denote the average value of u over the ball of radius r around x. Then

$$I(x,r) = \frac{1}{\alpha_n r^n} \int_{|y-x|\le r} u(y)\,dy = \frac{1}{\alpha_n r^n} \int_0^r \int_{|y-x|=t} u(y)\,dS_y\,dt$$

and we can use the mean value property for spheres to get

$$I(x,r) = \frac{1}{\alpha_n r^n} \int_0^r n\alpha_n t^{n-1} u(x)\,dt = \frac{nu(x)}{r^n} \int_0^r t^{n-1}\,dt = u(x).$$

2. *Suppose that $A \subset \mathbb{R}^n$ is a bounded region and that $u\colon \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$\Delta u(x) = \lambda u(x) \quad \text{when } x \in A, \qquad u(x) = 0 \quad \text{when } x \in \partial A.$$

Show that u must be identically zero, if $\lambda \ge 0$. Hint: write down Green's identity for the integral of $u\Delta u$ and then simplify.

- Following the hint, let us first use Green's identity to write

$$\int_A u\Delta u\,dx = -\int_A \nabla u \cdot \nabla u\,dx + \int_{\partial A} u\nabla u \cdot \boldsymbol{n}\,dS.$$

Since $\Delta u = \lambda u$ within A and $u = 0$ on the boundary, it easily follows that

$$\int_A \lambda u^2\,dx = -\int_A |\nabla u|^2\,dx = -\sum_{i=1}^n \int_A u_{x_i}^2\,dx.$$

Here, the left hand side is non-negative, whereas the right hand side is non-positive. This means they must both be zero, so each u_{x_i} is identically zero and u is constant. Given that $u = 0$ on the boundary, we conclude that $u = 0$ at all points.

3. *Show that $H' = \delta$ in the sense of distributions, where H is the Heaviside function*

$$H(x) = \left\{ \begin{array}{ll} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{array} \right\}.$$

- Assuming that φ is a smooth function of compact support, we get

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_{-\infty}^{\infty} H(x)\varphi'(x)\,dx.$$

In view of the definition of H, this also implies that

$$\langle H', \varphi \rangle = -\int_0^{\infty} \varphi'(x)\,dx = \Big[-\varphi(x)\Big]_{x=0}^{\infty} = \varphi(0) = \langle \delta, \varphi \rangle.$$

4. *We say that $u\colon \mathbb{R}^n \to \mathbb{R}$ is subharmonic, if $\Delta u(x) \geq 0$ for all $x \in \mathbb{R}^n$. Show that u^2 is subharmonic whenever u is harmonic.*

- To show that $v = u^2$ is subharmonic, we note that $v_{x_i} = 2uu_{x_i}$ and that

$$v_{x_i x_i} = 2u_{x_i}u_{x_i} + 2uu_{x_i x_i} \quad \Longrightarrow \quad \Delta v = 2|\nabla u|^2 + 2u\Delta u = 2|\nabla u|^2 \geq 0.$$

5. *Find all harmonic functions $u(x,y)$ which have the form $u(x,y) = F(x/y)$.*

- Differentiating $u(x,y) = F(x/y)$ twice, one finds that

$$u_x = F'(x/y)\cdot(1/y) \quad \Longrightarrow \quad u_{xx} = F''(x/y)\cdot(1/y)^2,$$

while a similar computation gives

$$u_y = F'(x/y)\cdot(-x/y^2) \quad \Longrightarrow \quad u_{yy} = F''(x/y)\cdot(-x/y^2)^2 + F'(x/y)\cdot(2x/y^3).$$

In particular, $u(x,y)$ is harmonic if and only if

$$0 = u_{xx} + u_{yy} = F''(x/y)\cdot\frac{x^2+y^2}{y^4} + F'(x/y)\cdot\frac{2x}{y^3}\,.$$

Multiplying through by y^2 and setting $z = x/y$, one can write this equation as

$$F''(z)(z^2+1) + 2zF'(z) = 0 \quad \Longrightarrow \quad F''(z) = -\frac{2zF'(z)}{z^2+1}\,.$$

Since this is a separable ODE, we can now separate variables to get

$$\begin{aligned} \frac{F''(z)}{F'(z)} = -\frac{2z}{z^2+1} \quad &\Longrightarrow \quad \log F'(z) = C_0 - \log(z^2+1) \\ &\Longrightarrow \quad F'(z) = C_1(z^2+1)^{-1}. \end{aligned}$$

In particular, $F(z) = C_1 \arctan z + C_2$ and thus $u(x,y) = C_1 \arctan(x/y) + C_2$.

6. *Find the unique solution $u(x, y)$ of the Dirichlet problem*

$$u_{xx} + u_{yy} = 0 \quad \textit{when } x^2 + y^2 < a^2, \qquad u(x, y) = g(x, y) \quad \textit{when } x^2 + y^2 = a^2.$$

- Consider, more generally, the n-dimensional Dirichlet problem

$$\Delta u(x) = 0 \quad \text{when } |x| < r, \qquad u(x) = g(x) \quad \text{when } |x| = r. \tag{1}$$

Changing variables by $w(x) = u(rx)$, one finds that w satisfies

$$\Delta w(x) = 0 \quad \text{when } |x| < 1, \qquad w(x) = g(rx) \quad \text{when } |x| = 1 \tag{2}$$

if and only if u satisfies (1). We already know that the solution of (2) is given by

$$w(x) = \frac{1 - |x|^2}{n\alpha_n} \int_{|y|=1} \frac{g(ry)}{|x - y|^n} \, dS_y,$$

the standard Poisson formula. In particular, the solution of (1) is given by

$$u(x) = w(x/r) = \frac{1 - |x/r|^2}{n\alpha_n} \int_{|y|=1} \frac{g(ry)}{|x/r - y|^n} \, dS_y$$

and we can now change variables by $z = ry$ to conclude that

$$u(x) = \frac{r^2 - |x|^2}{n\alpha_n r^2} \int_{|z|=r} \frac{r^n g(z)}{|x - z|^n} \frac{dS_z}{r^{n-1}} = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{|z|=r} \frac{g(z)}{|x - z|^n} \, dS_z.$$

7. *Solve $u_{xx} + u_{yy} = 1$ in the annulus $a^2 \leq x^2 + y^2 \leq b^2$ subject to zero Dirichlet boundary conditions. Hint: looking for radial solutions, one ends up with an ODE.*

- Let us look for radial solutions, say $u(x, y) = F(x^2 + y^2)$, in which case

$$\begin{aligned} u_x &= 2xF'(x^2 + y^2) &\implies\quad u_{xx} &= 2F'(x^2 + y^2) + (2x)^2 F''(x^2 + y^2), \\ u_y &= 2yF'(x^2 + y^2) &\implies\quad u_{yy} &= 2F'(x^2 + y^2) + (2y)^2 F''(x^2 + y^2). \end{aligned}$$

Adding these two equations and setting $z = x^2 + y^2$ for convenience, we now get

$$\begin{aligned} u_{xx} + u_{yy} = 1 \quad &\iff \quad 4F'(z) + 4zF''(z) = 1 \quad \iff \quad 4zF'(z) = z + C_0 \\ &\iff \quad F'(z) = \frac{1}{4} + \frac{C_1}{z} \quad \iff \quad F(z) = \frac{z}{4} + C_1 \log z + C_2. \end{aligned}$$

For the boundary conditions to hold, $F(z)$ should vanish when $z = a^2, b^2$ and so

$$\frac{a^2}{4} + 2C_1 \log a + C_2 = 0 = \frac{b^2}{4} + 2C_1 \log b + C_2.$$

Solving for C_1 and C_2, one finds that

$$C_1 = \frac{b^2 - a^2}{8\log(a/b)}, \qquad C_2 = -\frac{a^2}{4} - 2C_1 \log a.$$

In particular, the desired solution is given by

$$u(x,y) = \frac{x^2 + y^2}{4} + \frac{(b^2 - a^2)\log(x^2 + y^2)}{8\log(a/b)} + \frac{a^2 \log b - b^2 \log a}{4\log(a/b)}.$$

8. *Given a bounded region $A \subset \mathbb{R}^n$, show that the Dirichlet problem*

$$\Delta u(x) = f(x) \quad \textit{when } x \in A, \qquad u(x) = g(x) \quad \textit{when } x \in \partial A$$

has at most one solution. Give one proof using the maximum principle and one using Green's first identity. Hint: if u, v are solutions, then $w = u - v$ is harmonic, so its min/max values are attained on ∂A; use Green's identity for the integral of $w\Delta w$.

- First, we use the maximum principle. If u, v are both solutions, then $w = u - v$ is harmonic and also zero on the boundary. Moreover, a harmonic function attains both its min and its max on the boundary, so $w = 0$ at all points and $u = v$ at all points.

- We now give a proof using Green's identity. If u, v are both solutions, then $w = u - v$ is harmonic and also zero on the boundary. By Green's identity then, we have

$$\int_A w\Delta w \, dx = -\int_A \nabla w \cdot \nabla w \, dx + \int_{\partial A} w \nabla w \cdot \boldsymbol{n} \, dS,$$

where both the leftmost and the rightmost integrals are zero. This implies

$$\int_A \nabla w \cdot \nabla w \, dx = 0 \quad \Longrightarrow \quad \sum_{i=1}^n \int_A w_{x_i}^2 \, dx = 0,$$

so w is constant. Given that $w = 0$ on the boundary, we get $w = 0$ at all points.

9. *Suppose u is harmonic in the unit disc $x^2 + y^2 \leq 1$ and such that $u(x,y) = x^2$ on the boundary $x^2 + y^2 = 1$. Determine the value of u at the origin.*

- The value of u at the origin is equal to the mean value over any circle (sphere) around the origin. Looking at the circle of radius 1 around the origin, we now find

$$\begin{aligned} u(0,0) = \frac{1}{2\pi} \int_{x^2+y^2=1} u(x,y) \, dS &= \frac{1}{2\pi} \int_0^{2\pi} \cos^2\theta \, d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} [1 + \cos 2\theta] \, d\theta = \frac{1}{2}. \end{aligned}$$

10. *Find the unique bounded solution of the Laplace equation* $u_{xx} + u_{yy} = 0$ *in the upper half plane* $y \geq 0$ *subject to the boundary condition* $u(x, 0) = \operatorname{sign} x$.

- According to the Poisson formula, the unique bounded solution is given by

$$\begin{aligned} u(x,y) &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u(z,0)\,dz}{(x-z)^2 + y^2} \\ &= \frac{y}{\pi} \int_{0}^{\infty} \frac{dz}{(x-z)^2 + y^2} - \frac{y}{\pi} \int_{-\infty}^{0} \frac{dz}{(x-z)^2 + y^2}. \end{aligned}$$

Using the substitution $w = \frac{z-x}{y}$ to simplify these integrals, we now get

$$\begin{aligned} u(x,y) &= \frac{1}{\pi} \int_{-x/y}^{\infty} \frac{dw}{w^2+1} - \frac{1}{\pi} \int_{-\infty}^{-x/y} \frac{dw}{w^2+1} \\ &= \left[\frac{\arctan w}{\pi} \right]_{-x/y}^{\infty} - \left[\frac{\arctan w}{\pi} \right]_{-\infty}^{-x/y}. \end{aligned}$$

In particular, we get

$$u(x,y) = \frac{\pi/2}{\pi} - \frac{2\arctan(-x/y)}{\pi} + \frac{-\pi/2}{\pi} = \frac{2\arctan(x/y)}{\pi}.$$