**PDEs, Homework #3**
**Solutions**

**1.** *Use Hölder's inequality to show that the solution of the heat equation*

$$u_t = ku_{xx}, \qquad u(x,0) = \varphi(x) \tag{HE}$$

*goes to zero as $t \to \infty$, if $\varphi$ is continuous and bounded with $\varphi \in L^p$ for some $p \geq 1$. Hint: you will need to compute the $L^q$ norm of the heat kernel for some $q \geq 1$.*

- The solution of the initial value problem (HE) is given by the formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t) \cdot \varphi(y)\, dy,$$

where $S$ is the heat kernel. According to Hölder's inequality then, we have

$$|u(x,t)| \leq ||S(x-y,t)||_q \cdot ||\varphi(y)||_p$$

whenever $\frac{1}{p} + \frac{1}{q} = 1$. To compute the $L^q$ norm of the heat kernel, we write

$$\begin{aligned} ||S(x-y,t)||_q^q &= \int_{-\infty}^{\infty} S(x-y,t)^q\, dy \\ &= \frac{1}{(4k\pi t)^{q/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt} \cdot q\right) dy \end{aligned}$$

and we make an obvious change of variables to get

$$||S(x-y,t)||_q^q = \frac{(4kt/q)^{1/2}}{(4k\pi t)^{q/2}} \int_{-\infty}^{\infty} e^{-z^2} dz = \frac{(4k\pi t/q)^{1/2}}{(4k\pi t)^{q/2}} = Ct^{\frac{1-q}{2}}.$$

If $p > 1$, then $q = \frac{p}{p-1} > 1$ and the last equation gives

$$\lim_{t\to\infty} ||S(x-y,t)||_q = \lim_{t\to\infty} Ct^{\frac{1-q}{2q}} = 0.$$

If $p = 1$, on the other hand, then $q = \infty$ and we have

$$S(x-y,t) = \frac{1}{\sqrt{4k\pi t}} \cdot \exp\left(-\frac{(x-y)^2}{4kt}\right) \leq \frac{1}{\sqrt{4k\pi t}}.$$

Taking the limit as $t \to \infty$, we reach the same conclusion as before, namely

$$\lim_{t\to\infty} ||S(x-y,t)||_\infty = 0.$$

**2.** *Solve the Dirichlet problem for the heat equation on the half line. In other words, find the solution to* (HE) *when* $x > 0$ *and the boundary condition* $u(0,t) = 0$ *is imposed for all* $t \geq 0$. *Hint: argue as for the Neumann problem but use an odd extension.*

- Extend $\varphi$ to the whole real line in such a way that the extension is odd. Then the solution of the initial value problem on the whole real line is

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\varphi_{\text{ext}}(y)\,dy \\ &= \int_{0}^{\infty} S(x-y,t)\varphi(y)\,dy - \int_{-\infty}^{0} S(x-y,t)\varphi(-y)\,dy. \end{aligned}$$

Changing variables by $z = -y$ in the second integral, we now get

$$u(x,t) = \int_0^\infty \Big[S(x-z,t) - S(x+z,t)\Big]\varphi(z)\,dz.$$

**3.** *Use a substitution of the form* $v(x,t) = e^{at}u(x,t)$ *to solve the initial value problem*

$$u_t - ku_{xx} = bu, \qquad u(x,0) = \varphi(x)$$

*when* $b$ *is a constant. Hint: if* $a$ *is chosen suitably, then* $v$ *satisfies the heat equation.*

- Following the hint, let $v(x,t) = e^{at}u(x,t)$ and note that $v$ satisfies

$$\begin{aligned} v_t - kv_{xx} &= ae^{at}u + e^{at}u_t - ke^{at}u_{xx} \\ &= e^{at}(au + u_t - ku_{xx}) \\ &= e^{at}(a+b)u. \end{aligned}$$

If we take $a = -b$, then we end up with the heat equation and this implies

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\,dy \quad \Longrightarrow \quad u(x,t) = e^{bt}\int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\,dy.$$

**4.** *Show that the heat equation* (HE) *preserves positivity: if the initial temperature* $\varphi(x)$ *is non-negative for all* $x$, *then the temperature* $u(x,t)$ *is non-negative for all* $x,t$. *Show that the same is true for the associated Dirichlet problem on the interval* $[0,L]$. *Hint: the first part is easy; use the maximum principle to prove the second part.*

- When it comes to the initial value problem on the real line, it is clear that

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right)\cdot\varphi(y)\,dy$$

is non-negative, if $\varphi$ is. When it comes to the Dirichlet problem on $[0,L]$, we use the maximum principle to see that the minimum value of $u$ is attained on one of the three sides $x = 0$, $x = L$ and $t = 0$. Since $u = 0$ on the first two sides and $u = \varphi$ on the third, the minimum value is non-negative and so $u \geq \min u \geq 0$.

**5.** *Solve the initial value problem* (HE) *in the case* $\varphi(x) = e^{ax}$ *for some* $a \in \mathbb{R}$. *Hint: the solution is given by a messy integral; if you complete the square and use a suitable substitution, then you will end up with the standard integral involving* $e^{-z^2}$.

- The solution of the initial value problem is given by the formula

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt} + ay\right) dy \\ &= \frac{1}{\sqrt{4k\pi t}} \exp\left(-\frac{x^2}{4kt}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{4kt} + \frac{2xy}{4kt} + ay\right) dy. \end{aligned}$$

In order to simplify this formula, we now complete the square and we write

$$\begin{aligned} -\frac{y^2}{4kt} + \frac{2xy}{4kt} + ay &= -\frac{1}{4kt}\left(y^2 - 2y(x+2akt)\right) \\ &= -\frac{1}{4kt}\left(y - (x+2akt)\right)^2 + \frac{(x+2akt)^2}{4kt}. \end{aligned}$$

Changing variables by $z = \frac{y-x-2akt}{\sqrt{4kt}}$ and simplifying, we conclude that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4k\pi t}} \exp(ax + a^2kt) \int_{-\infty}^{\infty} \exp\left(-\frac{(y-x-2akt)^2}{4kt}\right) dy \\ &= \frac{1}{\sqrt{\pi}} \exp(ax + a^2kt) \int_{-\infty}^{\infty} e^{-z^2} dz \\ &= \exp(ax + a^2kt). \end{aligned}$$

**6.** *Prove uniqueness of solutions for the Neumann problem*

$$u_t = ku_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_x(0,t) = 0 = u_x(L,t). \tag{NP}$$

*Hint: if* $u, v$ *are both solutions, then* $E(t) = \int_0^L [u(x,t) - v(x,t)]^2\, dx$ *is decreasing.*

- Following the hint, set $w = u - v$ and consider the energy

$$E(t) = \int_0^L w(x,t)^2\, dx \quad \Longrightarrow \quad E'(t) = \int_0^L 2ww_t\, dx.$$

Since $w$ is itself a solution of the heat equation, one easily gets

$$E'(t) = \int_0^L 2kww_{xx}\, dx = -\int_0^L 2kw_x^2\, dx \le 0$$

using an integration by parts. In particular, $E(t)$ is decreasing and so

$$\int_0^L w(x,t)^2\, dx = E(t) \le E(0) = \int_0^L w(x,0)^2\, dx = 0.$$

Since the leftmost integral is non-negative, this implies $w(x,t) = 0$ at all points.

**7.** *Show that the solution to* (HE) *is even in $x$, if the initial datum $\varphi(x)$ is even.*

- To show that $w(x,t) = u(-x,t)$ is also a solution, we note that

  $$w_t(x,t) = u_t(-x,t) = ku_{xx}(-x,t) = kw_{xx}(x,t)$$

  and that $w(x,t)$ satisfies the initial condition

  $$w(x,0) = u(-x,0) = \varphi(-x) = \varphi(x).$$

  Since the initial value problem has a unique solution, this gives $u(-x,t) = u(x,t)$.

**8.** *Show that the non-homogeneous heat equation on the real line*

$$u_t - ku_{xx} = f(x,t), \qquad u(x,0) = \varphi(x)$$

*has a unique solution which is given by Duhamel's formula*

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-\tau)f(y,\tau)\,dy\,d\tau.$$

- If two solutions exist, then their difference satisfies the homogeneous problem subject to zero initial data, so it is zero by uniqueness. Thus, at most one solution exists and we need only check that Duhamel's formula gives a solution. Let us first focus on

  $$w(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-\tau)f(y,\tau)\,dy\,d\tau, \tag{1}$$

  namely the second term in Duhamel's formula. We note that

  $$w_t(x,t) = \int_{-\infty}^{\infty} S(x-y,0)f(y,t)\,dy + \int_0^t \int_{-\infty}^{\infty} S_t(x-y,t-\tau)f(y,\tau)\,dy\,d\tau \tag{2}$$

  and that the leftmost integral resembles the usual convolution formula with the heat kernel. In order to simplify this integral, we relate it to some other heat equation as follows. Suppose $t \geq 0$ is fixed and consider the initial value problem

  $$U_s(x,s) = kU_{xx}(x,s), \qquad U(x,0) = f(x,t).$$

  Using the explicit formula for the solution of this heat equation, we then get

  $$U(x,s) = \int_{-\infty}^{\infty} S(x-y,s)U(y,0)\,dy = \int_{-\infty}^{\infty} S(x-y,s)f(y,t)\,dy$$

  and we can substitute $s = 0$ to arrive at

  $$\int_{-\infty}^{\infty} S(x-y,0)f(y,t)\,dy = U(x,0) = f(x,t).$$

This allows us to simplify the leftmost integral in equation (2), so we find

$$\begin{aligned} w_t(x,t) &= f(x,t) + \int_0^t \int_{-\infty}^{\infty} S_t(x-y,t-\tau) f(y,\tau)\,dy\,d\tau \\ &= f(x,t) + \int_0^t \int_{-\infty}^{\infty} kS_{xx}(x-y,t-\tau) f(y,\tau)\,dy\,d\tau \\ &= f(x,t) + kw_{xx}(x,t). \end{aligned}$$

In particular, the rightmost integral in Duhamel's formula is a solution of

$$w_t - kw_{xx} = f(x,t), \qquad w(x,0) = 0$$

and the leftmost integral in Duhamel's formula is a solution of

$$v_t - kv_{xx} = 0, \qquad v(x,0) = \varphi(x).$$

Adding these two equations, we conclude that their sum $u = v + w$ satisfies

$$u_t - ku_{xx} = f(x,t), \qquad u(x,0) = \varphi(x).$$

**9.** *Show that the average temperature $T(t) = \int_0^L u(x,t)\,dx$ is conserved for each solution of the Neumann problem* (NP). *State the most general boundary conditions for which the average temperature is conserved.*

- According to the fundamental theorem of calculus, one has

$$T'(t) = \int_0^L u_t(x,t)\,dx = \int_0^L ku_{xx}(x,t)\,dx = \Big[ku_x(x,t)\Big]_{x=0}^L.$$

Thus, the average temperature is conserved, as long as $u_x(0,t) = u_x(L,t)$ at all times.