PDEs, Homework #2
Solutions

1. *Show that the solution $u(x,t)$ of the initial value problem*

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x) \tag{WE}$$

is even in x, if the initial data φ, ψ are even. Hint: show $u(-x,t)$ is also a solution.

- To show that $w(x,t) = u(-x,t)$ is also a solution, we note that

$$w_{tt}(x,t) = u_{tt}(-x,t) = c^2 u_{xx}(-x,t) = c^2 w_{xx}(x,t)$$

and that $w(x,t)$ satisfies the initial conditions

$$w(x,0) = u(-x,0) = \varphi(-x) = \varphi(x),$$
$$w_t(x,0) = u_t(-x,0) = \psi(-x) = \psi(x).$$

Since the initial value problem has a unique solution, this implies $u(x,t) = u(-x,t)$.

2. *Solve the Neumann problem for the wave equation on the half line. That is, find the solution to* (WE) *when $x > 0$ and the boundary condition $u_x(0,t) = 0$ is imposed for all $t \geq 0$. Hint: argue as for the Dirichlet problem but use an even extension.*

- Extend the initial data φ, ψ to the whole real line in such a way that the extension is even. Then the solution to the Cauchy problem on the whole real line is

$$u(x,t) = \frac{\varphi_{\text{ext}}(x-ct) + \varphi_{\text{ext}}(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds$$

and we need to express this in terms of φ, ψ. When $x > ct$, we get the usual formula

$$u(x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

because $x \pm ct$ are both positive. When $0 < x < ct$, only $x + ct$ is positive and so

$$u(x,t) = \frac{\varphi(ct-x) + \varphi(x+ct)}{2} + \frac{1}{2c}\left[\int_{x-ct}^{0} \psi(-s)\,ds + \int_0^{x+ct} \psi(s)\,ds\right].$$

Making the substitution $r = -s$ in the first integral, we conclude that

$$u(x,t) = \frac{\varphi(ct-x) + \varphi(ct+x)}{2} + \frac{1}{2c}\left[\int_0^{ct-x} \psi(r)\,dr + \int_0^{ct+x} \psi(r)\,dr\right].$$

3. *Find all solutions $u = u(x, y)$ of the second-order equation $u_{xx} + 4u_{xy} + 3u_{yy} = 0$.*

- First of all, let us factor the given PDE and write

$$0 = (\partial_x^2 + 4\partial_x\partial_y + 3\partial_y^2)u = (\partial_x + \partial_y)(\partial_x + 3\partial_y)u.$$

If we can find variables v, w such that $\partial_v = \partial_x + \partial_y$ and $\partial_w = \partial_x + 3\partial_y$, then

$$0 = \partial_v\partial_w u \quad \Longrightarrow \quad \partial_w u = F_1(w) \quad \Longrightarrow \quad u = F_2(w) + F_3(v).$$

To actually find the variables v and w, we note that

$$\partial_v = x_v\partial_x + y_v\partial_y, \qquad \partial_w = x_w\partial_x + y_w\partial_y$$

by the chain rule, while $\partial_v = \partial_x + \partial_y$ and $\partial_w = \partial_x + 3\partial_y$ by above. This gives

$$x_v = y_v = x_w = 1, \qquad y_w = 3$$

so we can let $x = v + w$ and $y = v + 3w$. Solving for v and w, we conclude that

$$y - x = 2w, \qquad 3x - y = 2v \quad \Longrightarrow \quad u = F(y - x) + G(3x - y).$$

4. *Show that the solution to the wave equation* (WE) *need not remain bounded at all times, even though it is initially bounded. Hint: take the initial data to be constant.*

- Suppose that $\varphi(x) = \psi(x) = 1$, for instance. Then the corresponding solution is

$$u(x, t) = \frac{1 + 1}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} ds = 1 + t$$

by d'Alembert's formula, so the solution does not remain bounded at all times.

5. *Solve the wave equation* (WE) *in the case that $\varphi(x) = x^2$ and $\psi(x) = x + 1$.*

- According to d'Alembert's formula, the solution is given by

$$u(x, t) = \frac{(x + ct)^2 + (x - ct)^2}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} (s + 1)\, ds.$$

When it comes to the integral on the right hand side, one easily finds that

$$\int_{x-ct}^{x+ct} (s + 1)\, ds = \frac{(x + ct)^2 - (x - ct)^2}{2} + 2ct = 2cxt + 2ct.$$

Using this fact and a little bit of algebra, we conclude that

$$u(x, t) = x^2 + (ct)^2 + xt + t.$$

6. *Suppose that $a < b$ and consider the Cauchy problem* (WE) *in the case that*

$$\varphi(x) = \psi(x) = \left\{ \begin{array}{cc} 1 & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{array} \right\}.$$

Compute the limit $\lim_{t\to\infty} u(x,t)$ for each fixed $x \in \mathbb{R}$.

- According to d'Alembert's formula, the solution is given by

$$u(x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

Since $x \in \mathbb{R}$ is fixed, we have $x \pm ct \to \pm\infty$ as $t \to \infty$, and this implies

$$\lim_{t\to\infty} u(x,t) = \frac{1}{2c}\int_{-\infty}^{\infty} \psi(s)\,ds = \frac{1}{2c}\int_a^b ds = \frac{b-a}{2c}.$$

7. *Solve the following non-homogeneous wave equation on the real line:*

$$u_{tt} - c^2 u_{xx} = t, \qquad u(x,0) = x^2, \qquad u_t(x,0) = 1.$$

- According to Duhamel's formula, the solution is given by

$$u(x,t) = \frac{(x+ct)^2 + (x-ct)^2}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} ds + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \tau\,dy\,d\tau.$$

When it comes to the rightmost integral, one easily finds that

$$\frac{1}{2c}\int_0^t \tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} dy\,d\tau = \int_0^t \tau(t-\tau)\,d\tau = \left[\frac{\tau^2 t}{2} - \frac{\tau^3}{3}\right]_{\tau=0}^{t} = \frac{t^3}{6}.$$

Using this fact and simplifying the remaining terms, we conclude that

$$u(x,t) = x^2 + (ct)^2 + t + \frac{t^3}{6}.$$

8. *Use the substitution $v(x,t) = e^{\lambda t}u(x,t)$ to solve the initial value problem*

$$u_{tt} - u_{xx} + 2\lambda u_t + \lambda^2 u = 0, \qquad u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x)$$

on the real line. Hint: you should find that v satisfies the wave equation $v_{tt} = v_{xx}$.

- Letting $v = e^{\lambda t}u$, we have $v_{xx} = e^{\lambda t}u_{xx}$ and also $v_t = \lambda e^{\lambda t}u + e^{\lambda t}u_t$, hence

$$\begin{aligned} v_{tt} - v_{xx} &= (\lambda^2 e^{\lambda t}u + 2\lambda e^{\lambda t}u_t + e^{\lambda t}u_{tt}) - e^{\lambda t}u_{xx} \\ &= e^{\lambda t}(\lambda^2 u + 2\lambda u_t + u_{tt} - u_{xx}). \end{aligned}$$

According to the given PDE, this expression is zero and so $v(x,t)$ satisfies

$$v_{tt} = v_{xx}, \qquad v(x,0) = \varphi(x), \qquad v_t(x,0) = \lambda\varphi(x) + \psi(x).$$

Solving this problem using d'Alembert's formula with $c = 1$, we now find

$$v(x,t) = \frac{\varphi(x+t) + \varphi(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \lambda\varphi(s) + \psi(s)\, ds.$$

Thus, the solution to the original problem is given by

$$u(x,t) = \frac{e^{-\lambda t}}{2}\left[\varphi(x+t) + \varphi(x-t) + \int_{x-t}^{x+t} \lambda\varphi(s) + \psi(s)\, ds\right].$$

9. *Consider the wave equation with damping $u_{tt} - c^2 u_{xx} + du_t = 0$ on the real line. Show that the energy is decreasing for all classical solutions of compact support, if $d > 0$.*

- Suppose u is a classical solution of compact support and consider the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t(x,t)^2 + c^2 u_x(x,t)^2\, dx.$$

Since u is $\mathcal{C}^2$ by assumption, the integrand is $\mathcal{C}^1$ and we have

$$\begin{aligned} E'(t) &= \int_{-\infty}^{\infty} u_t u_{tt} + c^2 u_x u_{xt}\, dx \\ &= \int_{-\infty}^{\infty} u_t(c^2 u_{xx} - du_t)\, dx + \int_{-\infty}^{\infty} c^2 u_x u_{xt}\, dx. \end{aligned}$$

Integrating by parts and using the fact that u_x vanishes at $x = \pm\infty$, we now get

$$\begin{aligned} E'(t) &= \int_{-\infty}^{\infty} u_t(c^2 u_{xx} - du_t)\, dx - \int_{-\infty}^{\infty} c^2 u_{xx} u_t\, dx \\ &= -d\int_{-\infty}^{\infty} u_t^2\, dx \leq 0. \end{aligned}$$

10. *Solve the Cauchy problem* (WE) *on the half line $x > 0$ when $\varphi(x) = \psi(x) = 1$ and the Dirichlet condition $u(0,t) = 0$ is imposed for all $t \geq 0$. Is your solution a classical one? Hint: there are different formulas for the cases $x > ct$ and $x \leq ct$.*

- When $x > ct$, the solution is given by d'Alembert's formula

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds = 1 + t.$$

When $0 < x < ct$, on the other hand, it is given by the formula

$$u(x,t) = \frac{\varphi(ct+x) - \varphi(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(s)\, ds = \frac{x}{c}.$$

Since these two expressions do not agree when $x = ct$, the solution is not continuous along the line $x = ct$, so it is certainly not a classical solution.

11. *Find the eigenfunctions and eigenvalues of $-\partial_x^2$ subject to Neumann boundary conditions on $[0, L]$. That is, find all nonzero functions $F(x)$ and all $\lambda \in \mathbb{R}$ such that*

$$-F''(x) = \lambda F(x), \qquad F'(0) = F'(L) = 0.$$

- First of all, we multiply the given ODE by $F(x)$ and then integrate to get

$$\lambda \int_0^L F(x)^2 \, dx = -\int_0^L F''(x)F(x) \, dx = \int_0^L F'(x)^2 \, dx.$$

Since the leftmost integral is positive, this implies $\lambda \geq 0$. If $\lambda = 0$, then we have

$$F''(x) = 0 \quad \Longrightarrow \quad F'(x) = 0 \quad \Longrightarrow \quad F(x) = C.$$

If $\lambda = m^2$ is positive, on the other hand, then we have

$$F''(x) = -m^2 F(x) \quad \Longrightarrow \quad F(x) = C_1 \sin(mx) + C_2 \cos(mx).$$

In this case, the boundary condition $F'(0) = 0$ gives

$$F'(x) = mC_1 \cos(mx) - mC_2 \sin(mx) \quad \Longrightarrow \quad 0 = mC_1,$$

while the boundary condition $F'(L) = 0$ gives

$$0 = F'(L) = -mC_2 \sin(mL) \quad \Longrightarrow \quad mL = k\pi$$

for some integer k. In particular, we have $m = k\pi/L$ for some integer k, so

$$F(x) = C_2 \cos(mx) = C_2 \cos\left(\frac{k\pi x}{L}\right), \qquad \lambda = m^2 = \left(\frac{k\pi}{L}\right)^2.$$

12. *Solve the wave equation $u_{tt} = 4u_{xx}$ on the interval $[0, \pi]$ subject to the conditions*

$$u(x, 0) = \cos x, \qquad u_t(x, 0) = 1, \qquad u(0, t) = 0 = u(\pi, t).$$

- The solution of the Dirichlet problem for the wave equation $u_{tt} = c^2 u_{xx}$ on $[0, L]$ is

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \sin \frac{n\pi ct}{L} + b_n \cos \frac{n\pi ct}{L} \right) \cdot \sin \frac{n\pi x}{L},$$

where the coefficients a_n, b_n are given by the formula

$$a_n = \frac{2}{n\pi c} \int_0^L \sin \frac{n\pi x}{L} \cdot \psi(x) \, dx, \qquad b_n = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} \cdot \varphi(x) \, dx.$$

In this case, we have $c = 2$ and $L = \pi$, so the coefficients a_n are

$$a_n = \frac{1}{n\pi} \int_0^\pi \sin(nx)\, dx = \frac{1 - \cos(n\pi)}{n^2\pi}\,.$$

To compute the coefficients b_n, one can integrate by parts to get

$$b_n = \frac{2}{\pi} \int_0^\pi \sin(nx) \cos x\, dx = -\frac{2n}{\pi} \int_0^\pi \cos(nx) \sin x\, dx$$

and then integrate by parts again to arrive at

$$b_n = \frac{2n}{\pi} \Big[\cos(nx) \cos x\Big]_0^\pi + \frac{2n^2}{\pi} \int_0^\pi \sin(nx) \cos x\, dx.$$

This standard argument expresses b_n in terms of b_n, so it actually gives

$$b_n = -\frac{2n}{\pi} \Big[\cos(n\pi) + 1\Big] + n^2 b_n \quad \Longrightarrow \quad b_n = \frac{2n[\cos(n\pi) + 1]}{\pi(n^2 - 1)}$$

whenever $n \neq 1$; the remaining case $n = 1$ can now be treated directly, as

$$b_1 = \frac{2}{\pi} \int_0^\pi \sin x \cos x\, dx = \frac{1}{\pi} \Big[(\sin x)^2\Big]_0^\pi = 0.$$