PDEs, Homework #1
Solutions

1. *Show that the linear change of variables*

$$v = x + y, \qquad w = x - y$$

transforms the equation $u_{xy} = 0$ *into the wave equation* $u_{vv} - u_{ww} = 0$.

- Differentiating with respect to x, we use the chain rule to get

$$u_x = u_v v_x + u_w w_x = u_v + u_w.$$

Differentiating with respect to y, we then similarly find that

$$\begin{aligned} u_{xy} &= (u_{vv} v_y + u_{vw} w_y) + (u_{wv} v_y + u_{ww} w_y) \\ &= u_{vv} - u_{vw} + u_{wv} - u_{ww} \\ &= u_{vv} - u_{ww}. \end{aligned}$$

2. *Which of the following PDEs are linear? Which of those are homogeneous?*

$$x^2 u_x + y^2 u_y = \sin(xy), \qquad e^x u_x + e^u u_y = 0, \qquad x u_{xx} + y u_{yy} = u_x.$$

- Only the first and the third are linear. Only the third is homogeneous.

3. *Find all solutions* $u = u(x, y)$ *of the equation* $u_{xy} = xy$.

- In this case, one may simply integrate the given PDE, namely

$$\begin{aligned} u_{xy} = xy \quad &\Longrightarrow \quad u_x = \int xy\,dy = \frac{xy^2}{2} + C_1(x) \\ &\Longrightarrow \quad u = \int \frac{xy^2}{2} + C_1(x)\,dx = \frac{x^2y^2}{4} + C_2(x) + C_3(y). \end{aligned}$$

4. *Find all separable solutions* $u = F(x)G(y)$ *of the equation* $x u_y = y u_x$.

- To say that $u = F(x)G(y)$ is a solution of $x u_y = y u_x$ is to say that

$$xF(x)G'(y) = yF'(x)G(y) \quad \Longleftrightarrow \quad \frac{G'(y)}{yG(y)} = \frac{F'(x)}{xF(x)}.$$

Here, we need a function of y to be equal to a function of x, so this implies

$$\frac{G'(y)}{yG(y)} = \frac{F'(x)}{xF(x)} = \lambda \quad \Longrightarrow \quad G'(y) = \lambda y G(y), \qquad F'(x) = \lambda x F(x)$$

for some $\lambda \in \mathbb{R}$. Once we now solve these two separable ODEs, we find that

$$G(y) = C_1 e^{\lambda y^2/2}, \qquad F(x) = C_2 e^{\lambda x^2/2} \quad \Longrightarrow \quad u(x, y) = C_3 e^{\mu x^2 + \mu y^2}.$$

5. *Find all separable solutions $u = F(x)G(y)H(z)$ of the equation $u_x - u_y + u_z = 0$.*

- To say that $u = F(x)G(y)H(z)$ is a solution of $u_x - u_y + u_z = 0$ is to say that

$$F'(x)G(y)H(z) - F(x)G'(y)H(z) + F(x)G(y)H'(z) = 0.$$

Dividing through by $F(x)G(y)H(z)$, we now arrive at the equation

$$\frac{F'(x)}{F(x)} - \frac{G'(y)}{G(y)} = -\frac{H'(z)}{H(z)}.$$

Here, the left hand side depends only on x, y and the right hand side only on z, so

$$\frac{F'(x)}{F(x)} - \frac{G'(y)}{G(y)} = -\frac{H'(z)}{H(z)} = \lambda.$$

Using the exact same argument as above, we can similarly argue that

$$\frac{F'(x)}{F(x)} = \frac{G'(y)}{G(y)} + \lambda \quad \Longrightarrow \quad \frac{F'(x)}{F(x)} = \frac{G'(y)}{G(y)} + \lambda = \mu.$$

This allows us to deduce three equations for the three unknowns, namely

$$F'(x) = \mu F(x), \qquad G'(y) = (\mu - \lambda)G(y), \qquad H'(z) = -\lambda H(z).$$

Solving these three simple ODEs, we conclude that

$$F(x) = C_1 e^{\mu x}, \qquad G(y) = C_2 e^{(\mu-\lambda)y}, \qquad H(z) = C_3 e^{-\lambda z}.$$

Thus, every separable solution of the given PDE has the form

$$u(x, y, z) = F(x)G(y)H(z) = C_4 e^{\mu x + (\mu-\lambda)y - \lambda z}.$$

6. *Find all solutions $u = u(x, t)$ of the equation $u_t + 2xtu_x = e^t$.*

- In this case, the characteristic equations are

$$\frac{dt}{ds} = 1, \qquad \frac{dx}{ds} = 2xt, \qquad \frac{du}{ds} = e^t$$

and we shall assume that $u(x_0, 0) = f(x_0)$. Then we have $t = s$ and also

$$\frac{dx}{x} = 2t\,ds = 2s\,ds \quad \Longrightarrow \quad \log x = s^2 + C \quad \Longrightarrow \quad x = x_0 e^{s^2},$$

so the rightmost characteristic equation gives

$$du = e^t\,ds = e^s\,ds \quad \Longrightarrow \quad u = e^s + C \quad \Longrightarrow \quad u = e^s - 1 + u_0.$$

Once we now recall that $s = t$, we may finally conclude that

$$u = e^t - 1 + f(x_0) = e^t - 1 + f(xe^{-t^2}).$$

7. *Find a function $f(x)$ for which the initial value problem*

$$u_x + u_y = 2xu, \qquad u(x,x) = f(x)$$

has no solutions and a function $f(x)$ for which it has infinitely many solutions.

- In this case, the characteristic equations are

$$\frac{dx}{ds} = 1, \qquad \frac{dy}{ds} = 1, \qquad \frac{du}{ds} = 2xu$$

and we shall assume that $u(0, y_0) = g(y_0)$. Then we have $x = s$ and $y = s + y_0$, so

$$\frac{du}{u} = 2x\,ds = 2s\,ds \quad \Longrightarrow \quad \log u = s^2 + C \quad \Longrightarrow \quad u = u_0 e^{s^2}.$$

Eliminating y_0 and s, we may thus conclude that

$$\begin{aligned} x = s = y - y_0 \quad &\Longrightarrow \quad u(x,y) = g(y_0)e^{x^2} = g(y-x)e^{x^2} \\ &\Longrightarrow \quad u(x,x) = g(0)e^{x^2}. \end{aligned}$$

Now, if $f(x)$ is a constant multiple of e^{x^2}, then the initial value problem will have an infinite number of solutions, as g can be any function for which $g(0)$ is equal to that constant. And if $f(x)$ is not a constant multiple of e^{x^2}, then no solutions exist.

8. *Show that the characteristic curves for the equation $yu_x - xu_y = 0$ are circles around the origin. Conclude that $u(x,0) = f(x)$ must be even for any solution u.*

- In this case, the characteristic equations are

$$\frac{dx}{ds} = y, \qquad \frac{dy}{ds} = -x, \qquad \frac{du}{ds} = 0.$$

The easiest way to solve this system of ODEs is to note that

$$x' = y, \qquad y' = -x \quad \Longrightarrow \quad x'' = y' = -x \quad \Longrightarrow \quad x'' + x = 0.$$

This gives $x = C_1 \sin s + C_2 \cos s$ and also $y = x' = C_1 \cos s - C_2 \sin s$, so

$$x^2 + y^2 = (C_1 \sin s + C_2 \cos s)^2 + (C_1 \cos s - C_2 \sin s)^2 = C_1^2 + C_2^2.$$

To show that $u(x,0) = f(x)$ must be even, we need to show that

$$u(x,0) = u(-x,0)$$

for all $x \in \mathbb{R}$. This follows trivially by above since the points $(\pm x, 0)$ lie on the same characteristic curve and since u is constant along this curve.

9. *Find all solutions* $u = u(x, y)$ *of the equation* $u_x + u_y + u = e^{y-x}$.

- In this case, the characteristic equations are

$$x' = 1, \qquad y' = 1, \qquad u' + u = e^{y-x}$$

so we have $x = s + x_0$ and $y = s + y_0$, while

$$u' + u = e^{y-x} = e^{y_0 - x_0}.$$

Note that this ODE is first-order linear with integrating factor e^s, namely

$$\begin{aligned} (e^s u)' = e^s e^{y_0 - x_0} \quad &\implies \quad e^s u = e^s e^{y_0 - x_0} + C \\ &\implies \quad e^s u = (e^s - 1) e^{y_0 - x_0} + u_0. \end{aligned}$$

Imposing a generic initial condition such as $u(x_0, 0) = f(x_0)$, we conclude that

$$\begin{aligned} y = s = x - x_0 \quad &\implies \quad u = (1 - e^{-s}) e^{-x_0} + f(x_0) e^{-s} \\ &\implies \quad u = (1 - e^{-y}) e^{y-x} + f(x - y) e^{-y}. \end{aligned}$$

10. *Find all solutions* $u = u(x, y, z)$ *of the initial value problem*

$$x u_x + 2y u_y + u_z = 3u, \qquad u(x, y, 0) = f(x, y).$$

- In this case, the characteristic equations are

$$x' = x, \qquad y' = 2y, \qquad z' = 1, \qquad u' = 3u$$

and the initial condition $u(x_0, y_0, 0) = f(x_0, y_0)$ implies that

$$x = x_0 e^s, \qquad y = y_0 e^{2s}, \qquad z = s, \qquad u = f(x_0, y_0) e^{3s}.$$

Once we now eliminate x_0, y_0 and s, we may conclude that

$$x_0 = x e^{-z}, \qquad y_0 = y e^{-2z} \quad \implies \quad u = f(x e^{-z}, y e^{-2z}) e^{3z}.$$

11. *Which of the following second-order equations are hyperbolic? elliptic? parabolic?*

$$u_{xx} - 2u_{xy} + u_{yy} = 0, \qquad 3u_{xx} + u_{xy} + u_{yy} = 0, \qquad u_{xx} - 5u_{xy} - u_{yy} = 0.$$

- The first equation is parabolic since $\Delta = 2^2 - 4 = 0$. The second equation is elliptic since $\Delta = 1^2 - 4 \cdot 3 < 0$. The third equation is hyperbolic since $\Delta = 5^2 + 4 > 0$.

12. *For which values of* a *is the equation* $a u_{xx} + a u_{xy} + u_{yy} = 0$ *elliptic?*

- The discriminant $\Delta = B^2 - 4AC = a^2 - 4a$ is negative if and only if

$$a(a - 4) < 0 \quad \iff \quad 0 < a < 4.$$