# Heat equation on the half line

▶ **Dirichlet**: Consider the Dirichlet problem for the heat equation

$$u_t = ku_{xx}, \qquad u(x,0) = \varphi(x), \qquad u(0,t) = 0$$

on the half line $x > 0$. To solve this problem, one extends $\varphi$ to the whole real line in such a way that the extension is odd and then solves the corresponding problem to get

$$u(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)] \cdot \varphi(y)\, dy.$$

▶ **Neumann**: Consider the Neumann problem for the heat equation

$$u_t = ku_{xx}, \qquad u(x,0) = \varphi(x), \qquad u_x(0,t) = 0$$

on the half line $x > 0$. To solve this problem, one extends $\varphi$ to the whole real line in such a way that the extension is even and then solves the corresponding problem to get

$$u(x,t) = \int_0^\infty [S(x-y,t) + S(x+y,t)] \cdot \varphi(y)\, dy.$$

# Heat equation on a closed interval

▶ **Dirichlet**: Consider the Dirichlet problem for the heat equation

$$u_t = ku_{xx}, \qquad u(x,0) = \varphi(x), \qquad u(0,t) = 0 = u(L,t)$$

on the closed interval $[0, L]$. Separation of variables leads to the solution

$$u(x,t) = \sum_{n=1}^\infty a_n \sin\left(\frac{n\pi x}{L}\right) \cdot \exp\left(-\frac{kn^2\pi^2 t}{L^2}\right), \tag{9.1}$$

where the coefficients $a_n$ are given by the Fourier coefficients

$$\varphi(x) = \sum_{n=1}^\infty a_n \sin\left(\frac{n\pi x}{L}\right) \quad \Longrightarrow \quad a_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) \cdot \varphi(x)\, dx. \tag{9.2}$$

Alternatively, one can proceed as before to extend $\varphi$ to the whole real line in such a way that the extension is odd and periodic of period $2L$.