Heat equation: initial value problem

Theorem 1 (Explicit formula). Consider the initial value problem on the real line

$$u_t = ku_{xx}, \qquad u(x,0) = \varphi(x). \tag{8.1}$$

If φ is bounded and continuous, then the unique solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t) \cdot \varphi(y)\, dy, \tag{8.2}$$

where $S(x,t)$ is the heat kernel defined by

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}} \cdot \exp\left(-\frac{x^2}{4kt}\right). \tag{8.3}$$

Lemma 2 (Heat kernel). The heat kernel (8.3) is a solution of the heat equation with

$$\int_{-\infty}^{\infty} S(x,t)\, dx = 1 \quad \text{for each } t > 0.$$

Theorem 3 (Maximum principle). Suppose u satisfies the heat equation $u_t = ku_{xx}$ in some closed, bounded region A in the xt-plane. Then the min/max values of u are attained on the boundary of A. Moreover, if $A = [0,L] \times [0,T]$ is a rectangle, then the min/max values of u are not attained on the top side $t = T$.

Corollary 4 (Uniqueness of solutions). Consider the most general Dirichlet problem

$$u_t - ku_{xx} = f(x,t), \qquad u(x,0) = \varphi(x), \qquad u(0,t) = g(t), \qquad u(L,t) = h(t)$$

on the closed interval $[0,L]$. Then this problem admits at most one solution.

Theorem 5 (Stability for the heat equation). If φ is bounded and continuous, then the initial value problem (8.1) has a unique solution which satisfies the estimate

$$||u(x,t)||_p \leq ||\varphi(x)||_p$$

for each $p \geq 1$. In particular, the heat equation (8.1) is well-posed in L^p for each $p \geq 1$.